THE FOURIER TRANSFORM

YU / HUGHES

SEPTEMBER 28-30, 2021

UMASS AMHERST

COMPLEX WAVES: REVIEW

BUILDING UP A SQUARE WAVE WITH SINES

https://en.wikipedia.org/wiki/User:LucasVB/Gallery#/media/File:Synthesis_square.gif

BUILDING UP A SQUARE WAVE WITH SINES

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BUILDING UP A SQUARE WAVE

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TWO REPRESENTATIONS OF SQUARE WAVE

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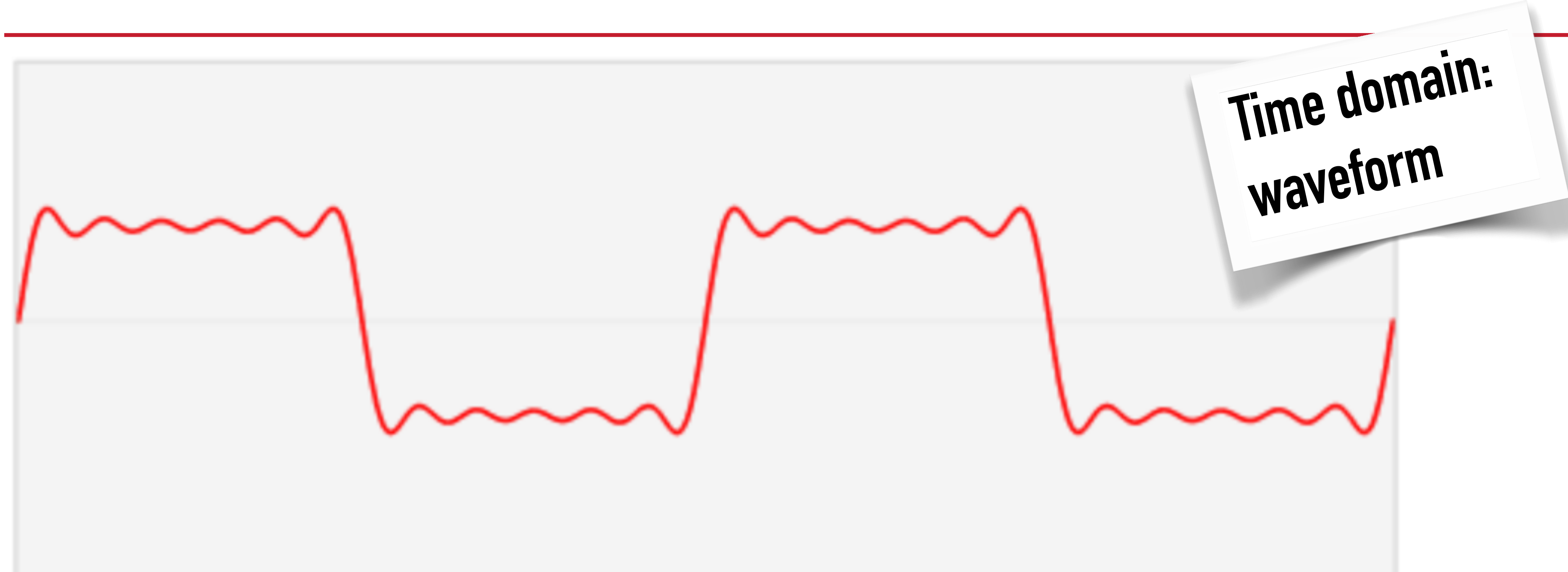
TWO REPRESENTATIONS OF SQUARE WAVE



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TWO REPRESENTATIONS OF SQUARE WAVE

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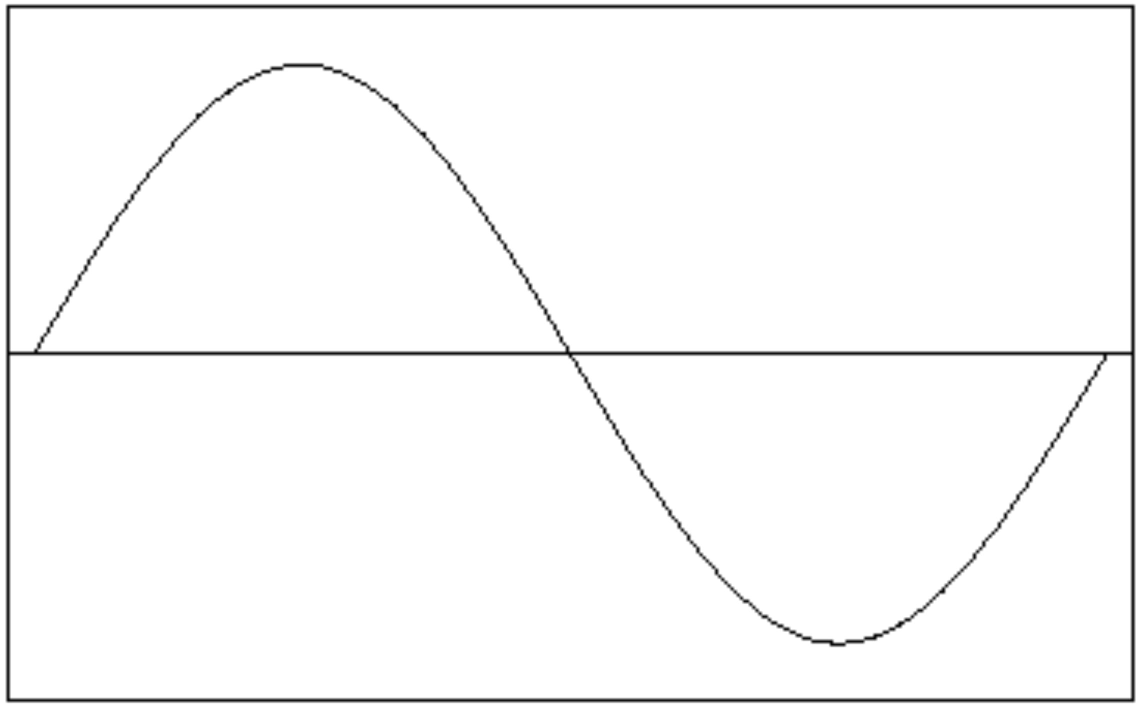
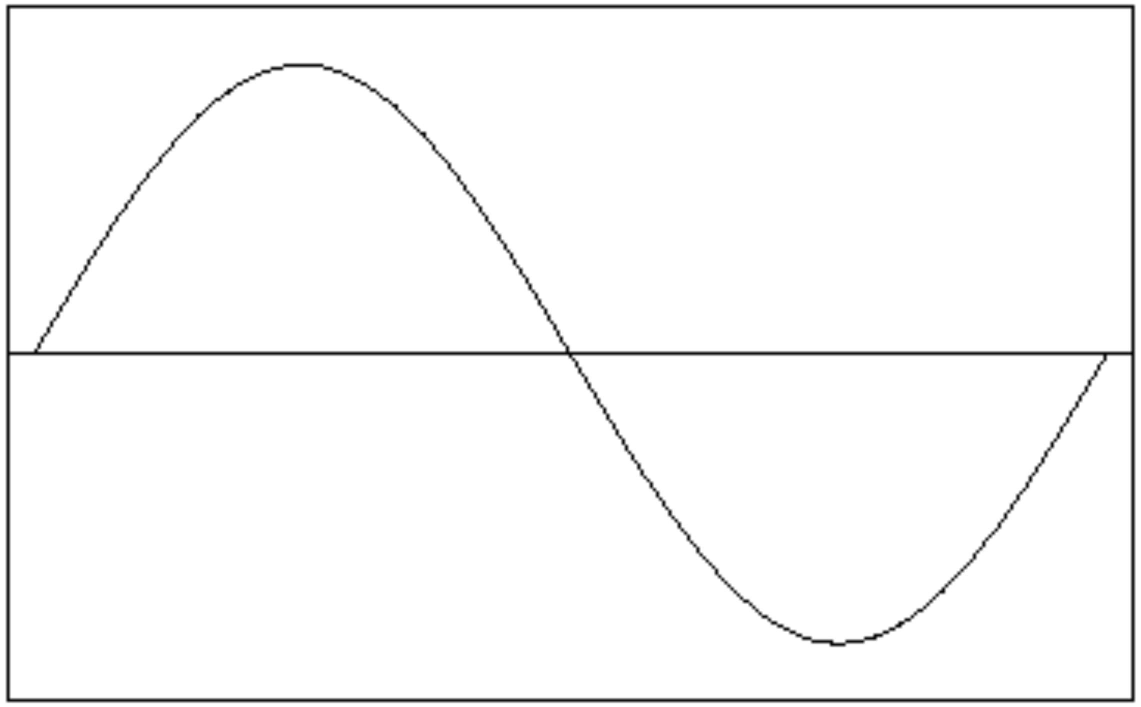
HARMONIC AMPLITUDES



https://sites.google.com/a/georgiasouthern.edu/julia-inozemtseva/teaching-math-animations-and-pics?tmpl=%2Fsystem%2Fapp%2Ftemplates%2Fprint%2F&showPrintDialog=1#TOC-Fourier-Series

HARMONIC AMPLITUDES



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EXERCISE: ADDITIVE SYNTHESIS 1

- **In a jupyter notebook, define the following component sinusoidal waves (use a sampling rate of 22050 Hz, and make the duration somewhere between 1 and 3 seconds):**
    - $cos(2\pi t)$, $sin(2\pi t)$
    - $cos(4\pi t)$, $sin(4\pi t)$
    - $cos(6\pi t)$, $sin(6\pi t)$
- ***What is the period of each of the component waves? What is the period of the complex wave created by adding together all the component waves? Why? Calculate the periods by hand, and also look visually at plots in jupyter notebook, at what happens as you add in additional terms in the Fourier series.***

EXERCISE II: ADDITIVE SYNTHESIS

- **Now do the same exercise, but for the following component waves (could this be a Fourier series?):**
    - $cos(\pi t)$
    - $sin(2\pi t)$
    - $cos(5\pi t)$
    - $sin(8\pi t)$
- ***What is the period of each of the component waves? What is the period of the complex wave created by adding together all the component waves? Why? Calculate the periods by hand, and also look visually at plots in jupyter notebook, at what happens as you add in additional terms in the Fourier series.***

FOURIER COEFFICIENTS

FOURIER SERIES

$$a_0 + \sum_{n=1}^{N} \left(a_n \cos(2\pi nt) + b_n \sin(2\pi nt) \right)$$

FOURIER COEFFICIENTS

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 2\pi ntdt$$

FOURIER SERIES INTEGRAL IDENTITIES

https://commons.wikimedia.org/wiki/File:Fourier_series_integral_identities.gif

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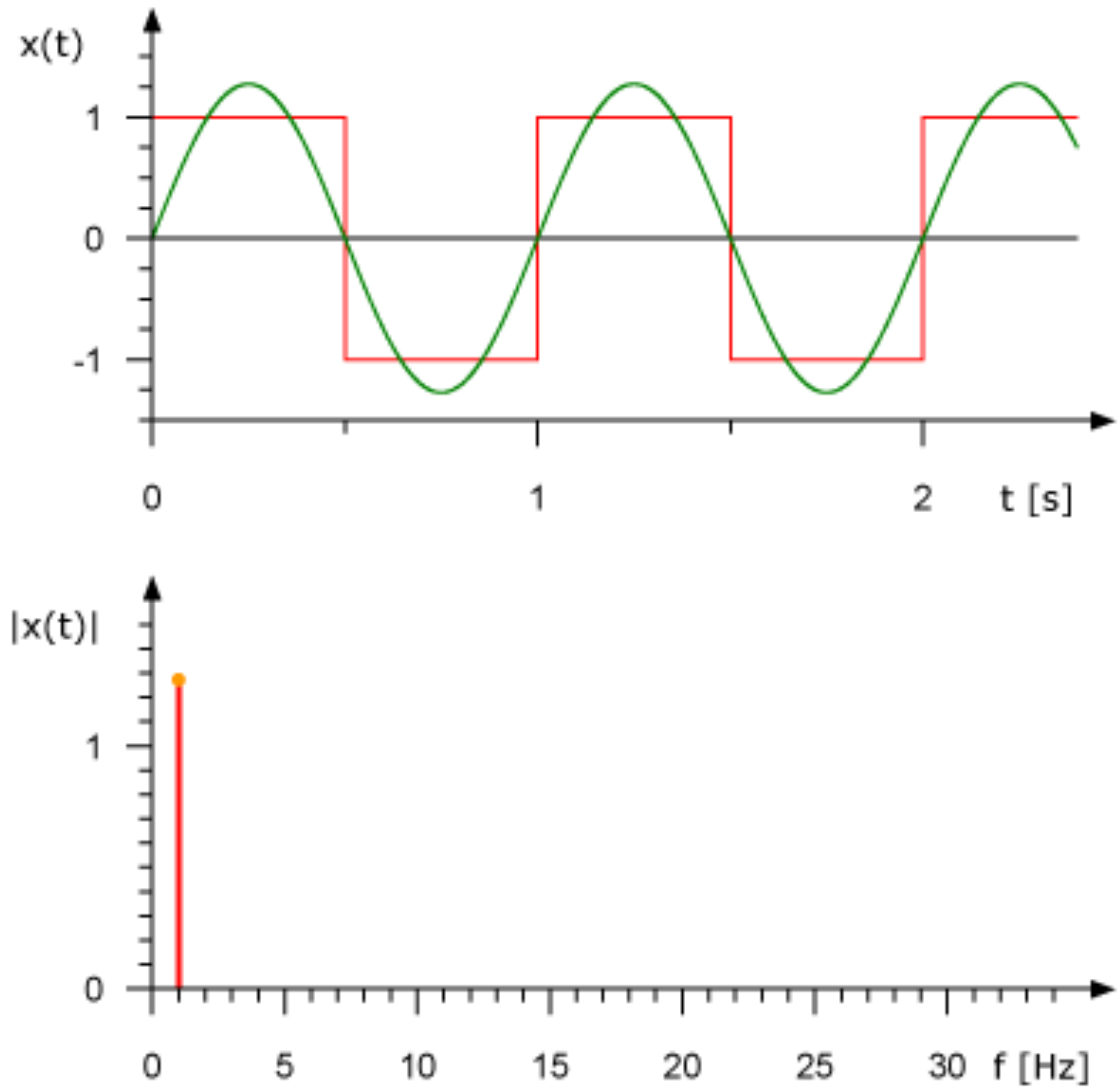
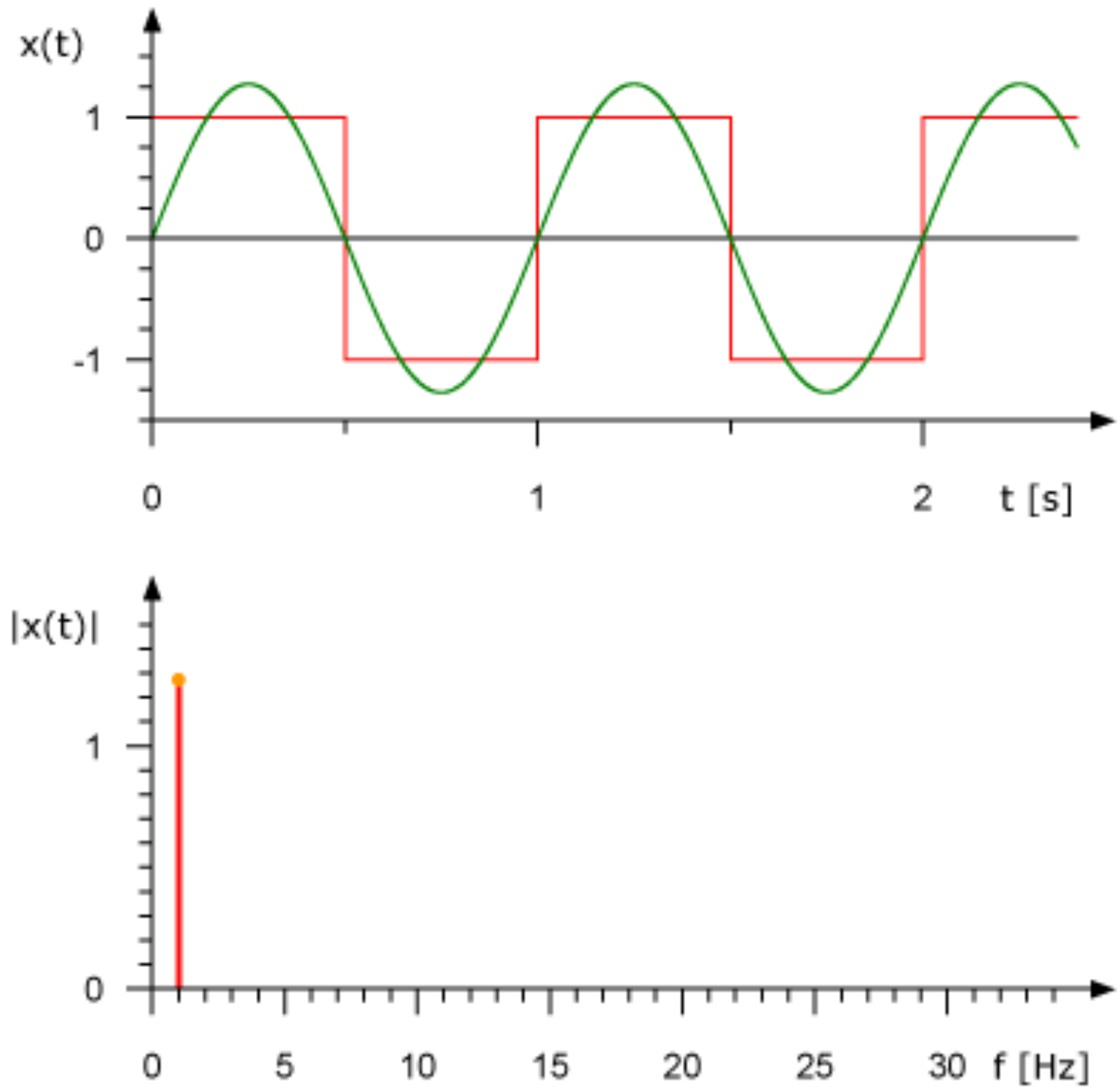
BACK TO SQUARE WAVES…



https://upload.wikimedia.org/wikipedia/commons/a/af/Fourier_synthesis_square_wave_animated.gif

BACK TO SQUARE WAVES…



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0TH TERM FOLDED INTO SUM

$$\sum_{n=0}^{N}(a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 2\pi ntdt$$

0TH TERM FOLDED INTO SUM

$$\sum_{n=0}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

How is the a_0 term here related to the a_0 when we separated out a_0 as a term?

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 2\pi ntdt$$

COMPLEX REPRESENTATION OF FOURIER SERIES

WHERE WE'RE HEADED…

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi int}$$

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi int} f(t)\,dt$$

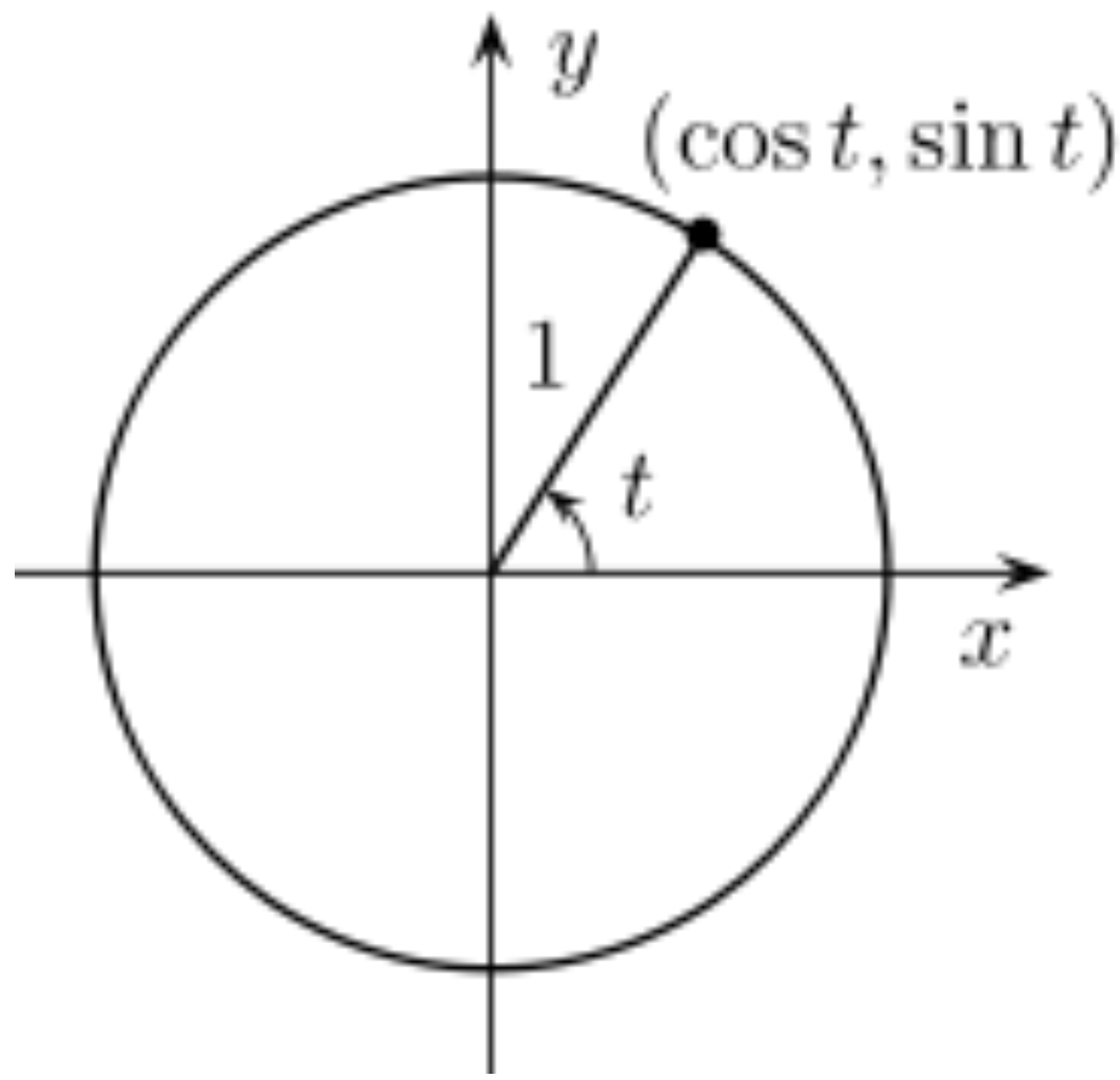
WHERE WE'RE HEADED...

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi i n t}$$

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi i n t} f(t)\,dt$$

See Who is Fourier Chs. 10,11
Osgood notes Sec 1.6.1

REMINDER: UNIT CIRCLE



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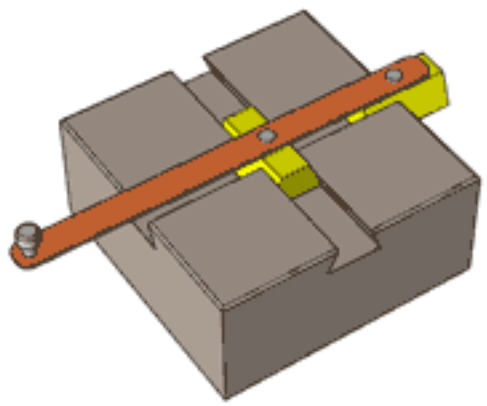
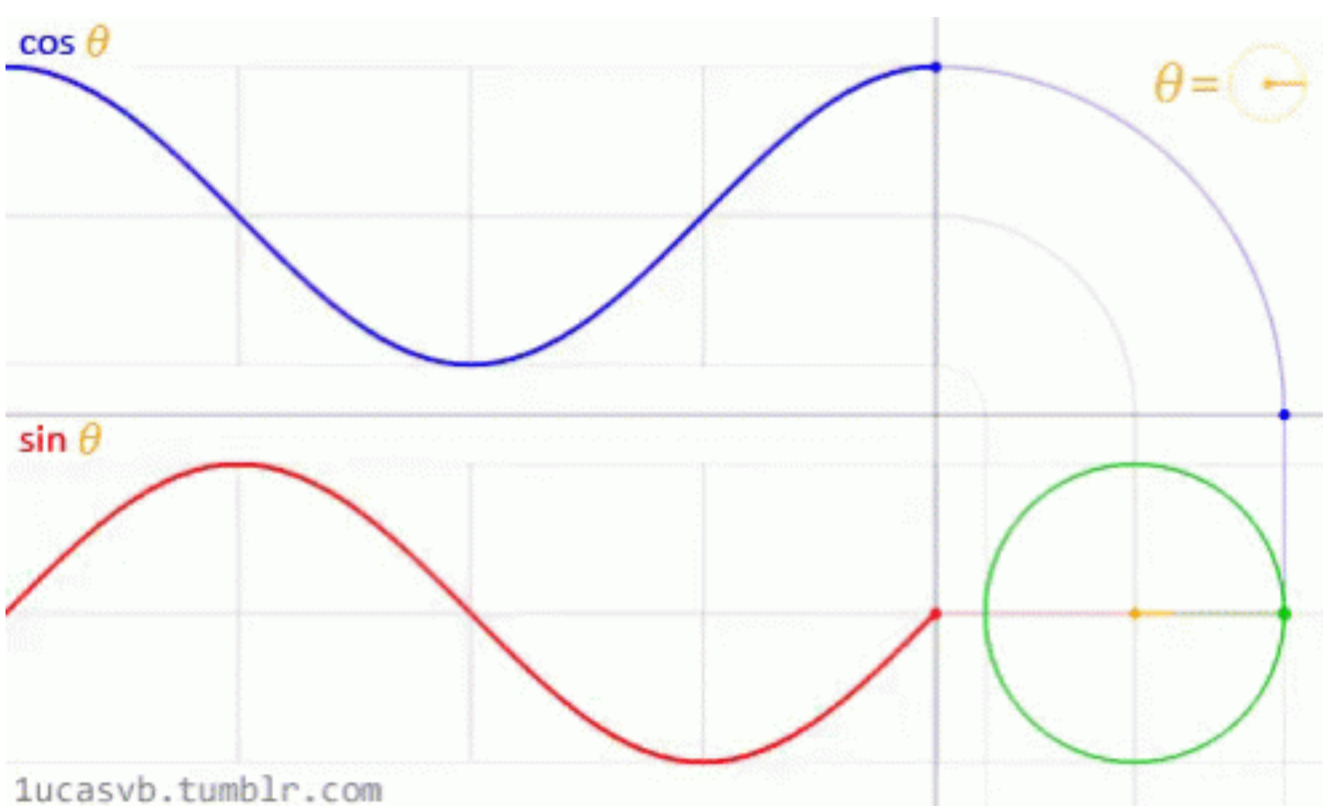
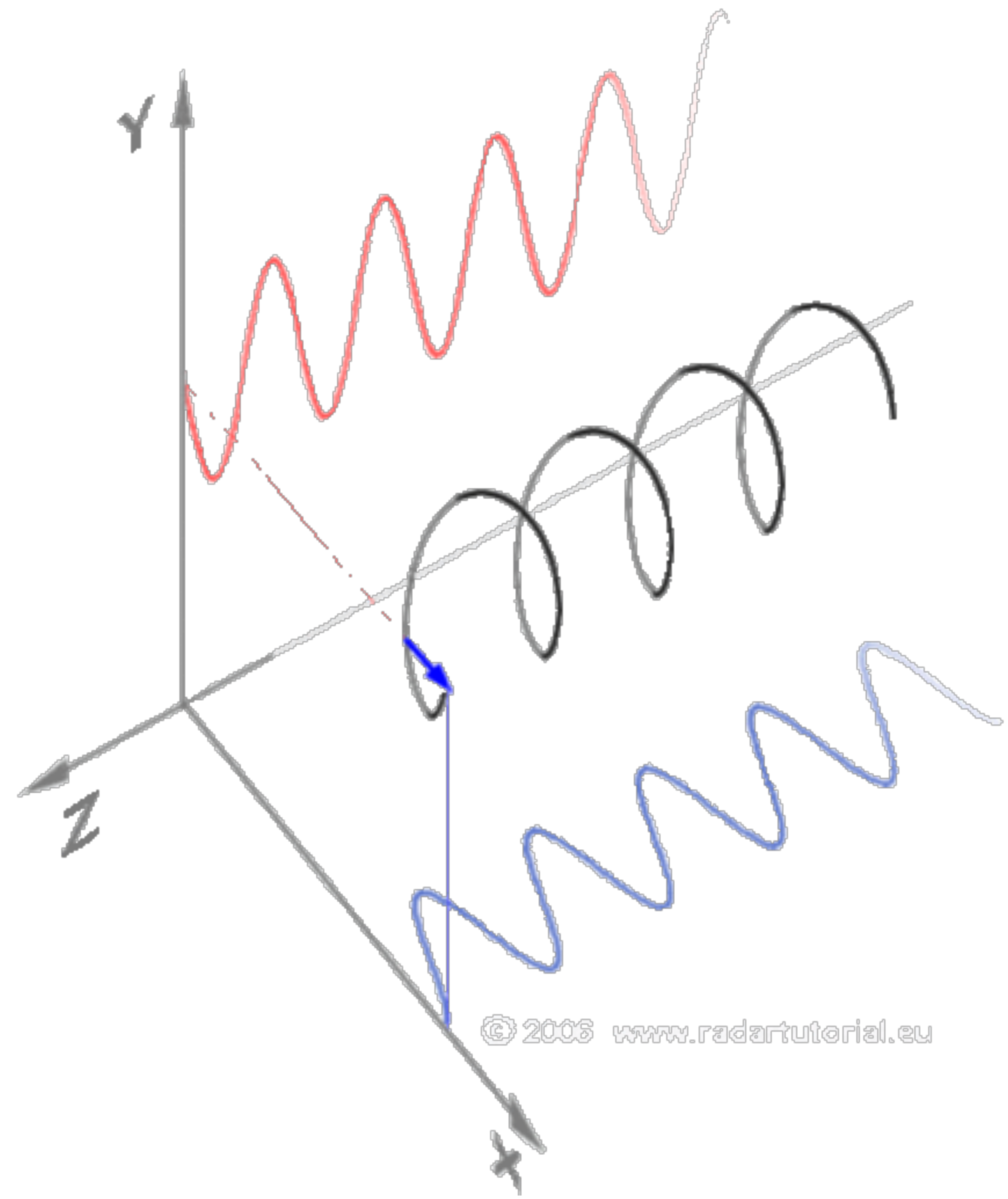
REMINDER: SINES AND COSINES







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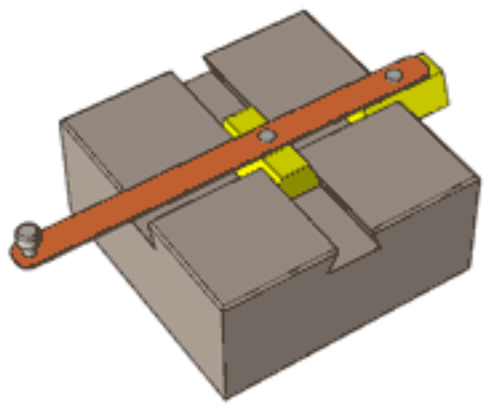

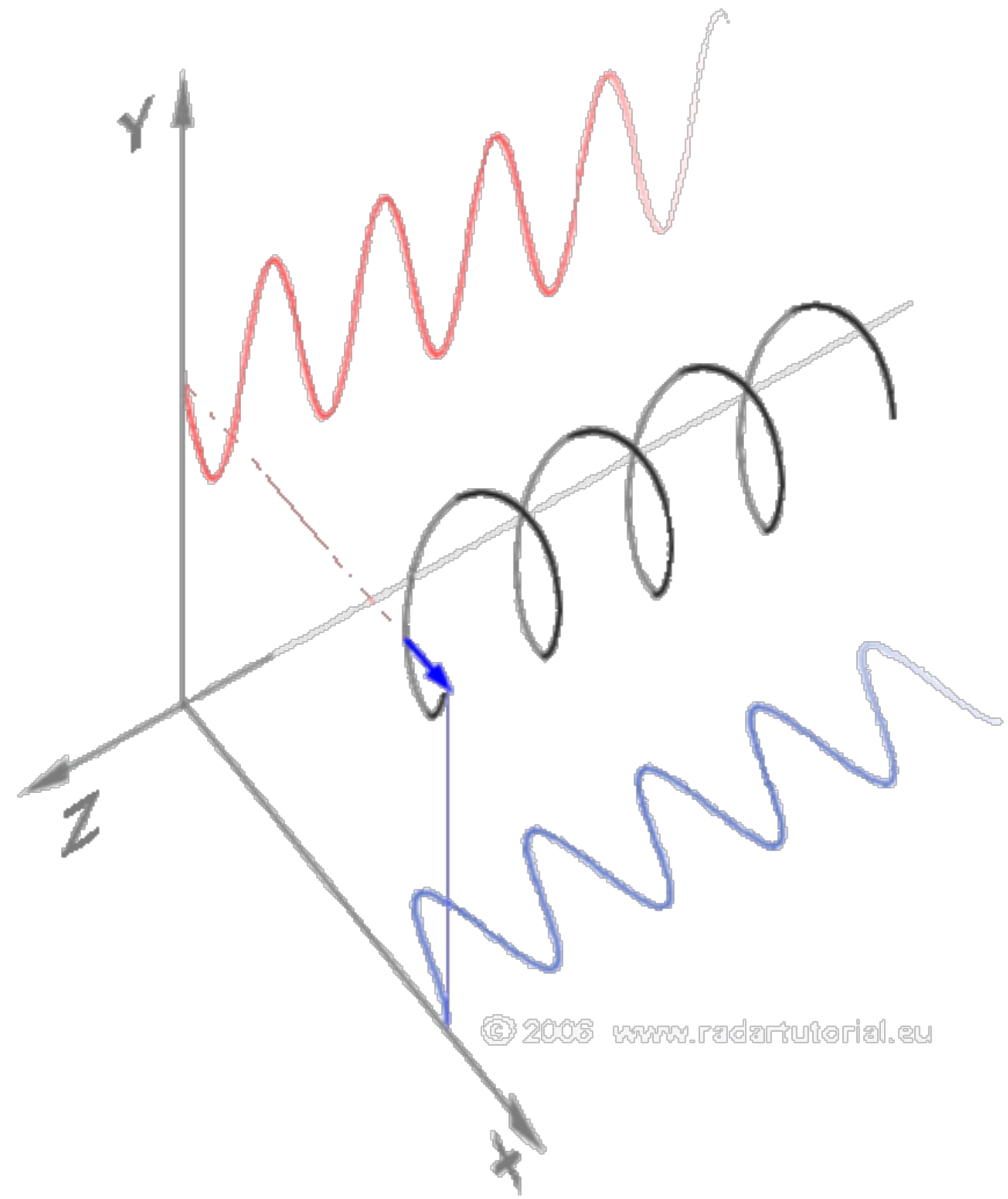

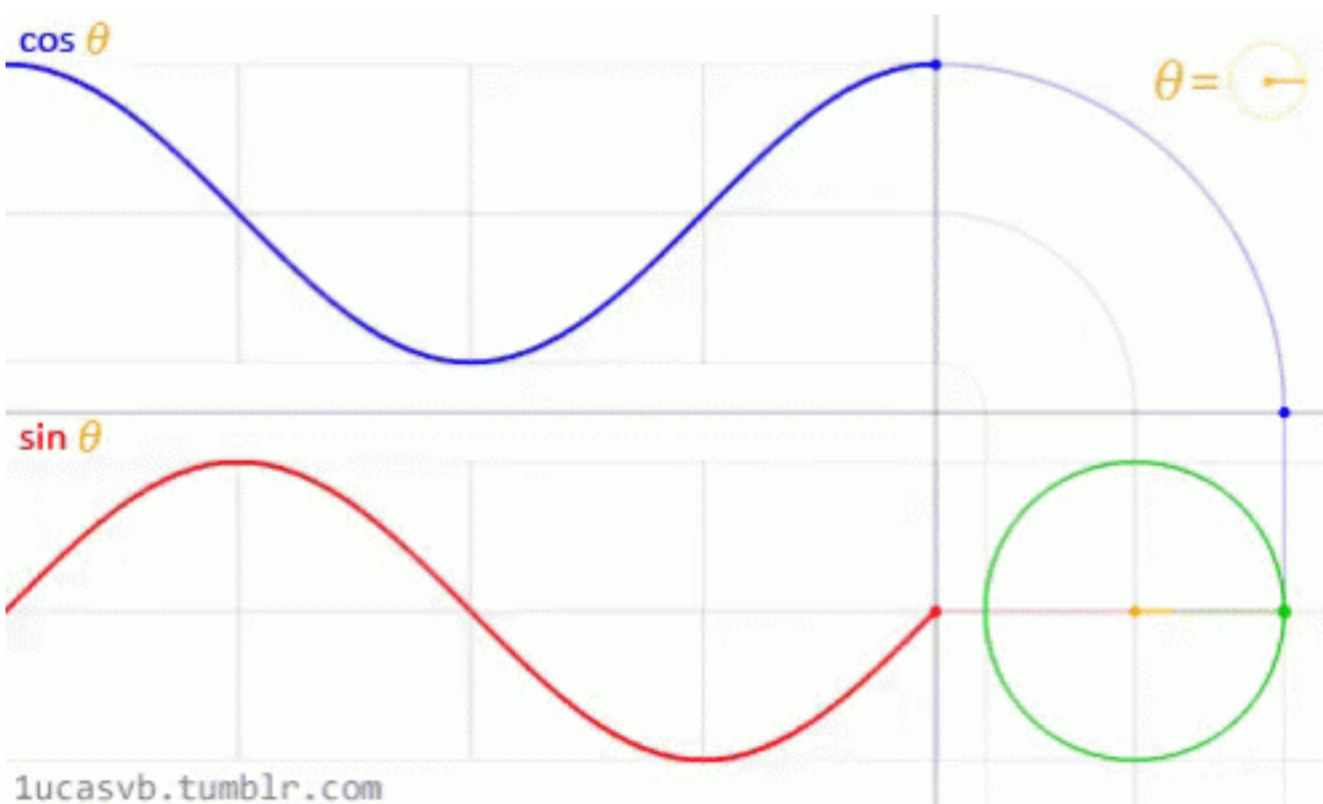

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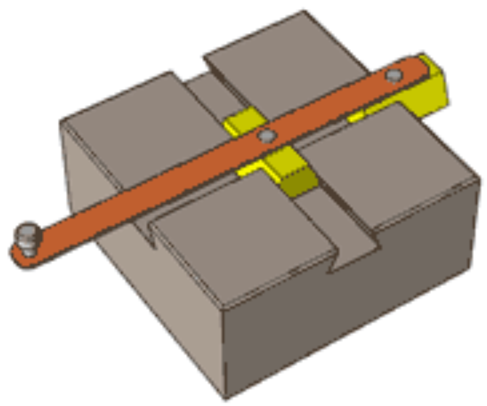

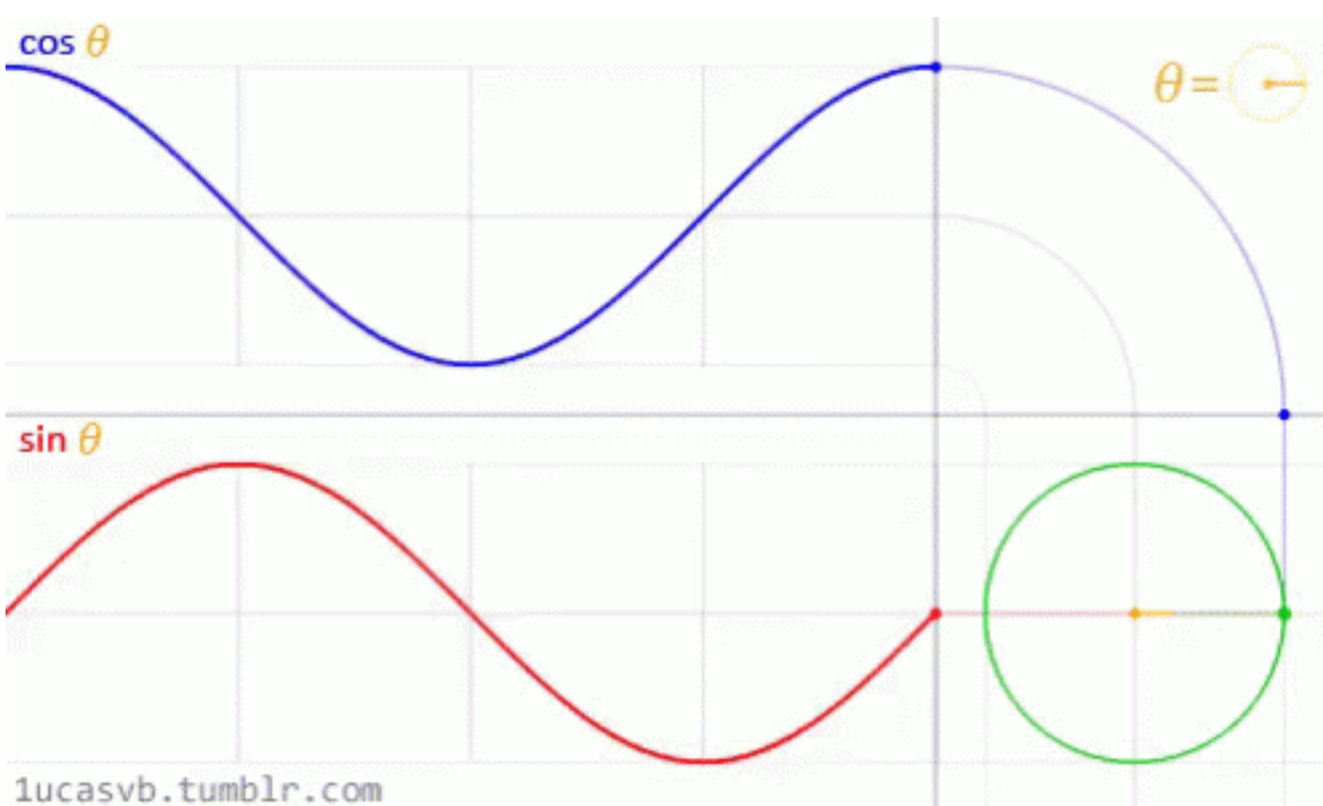

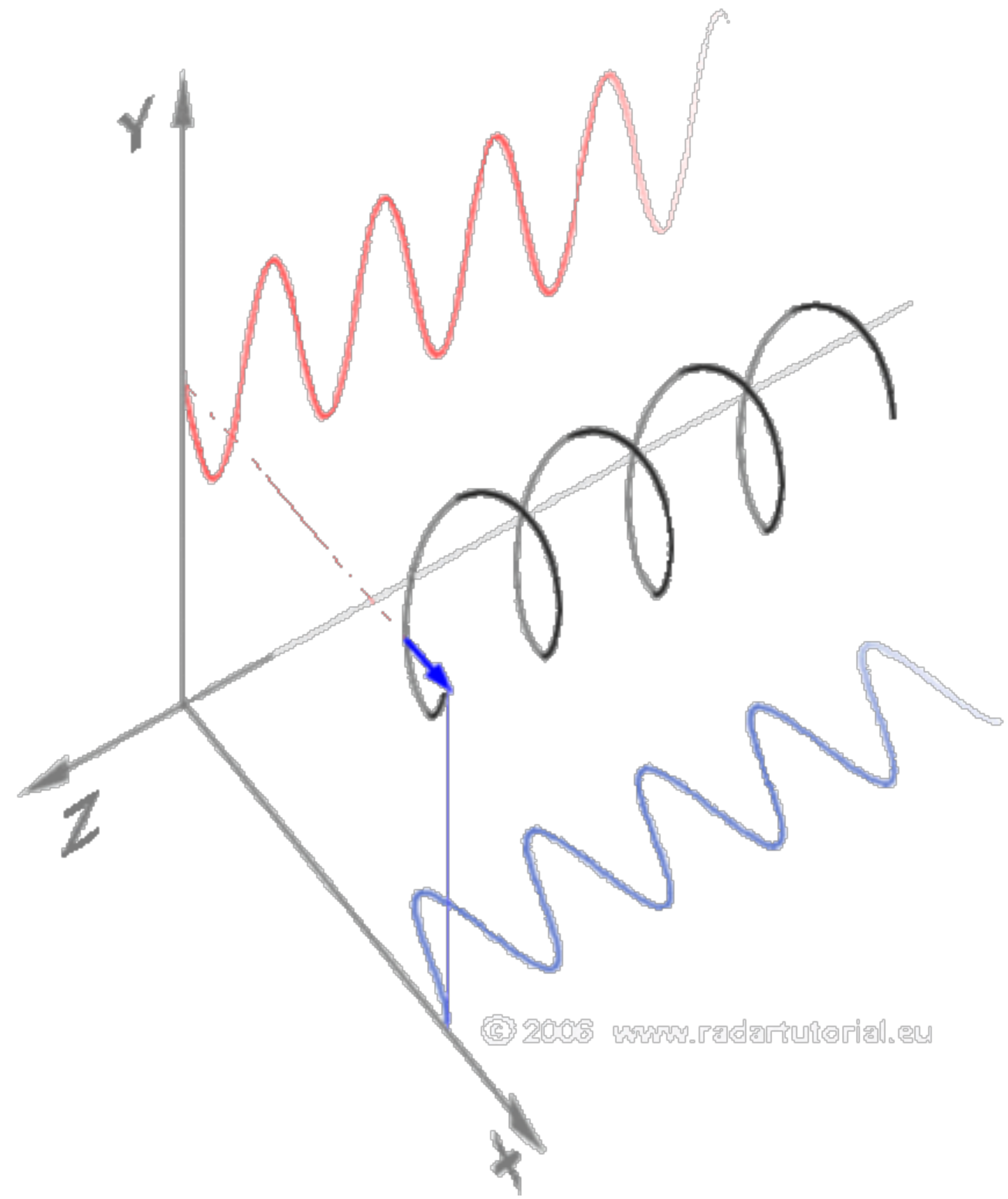

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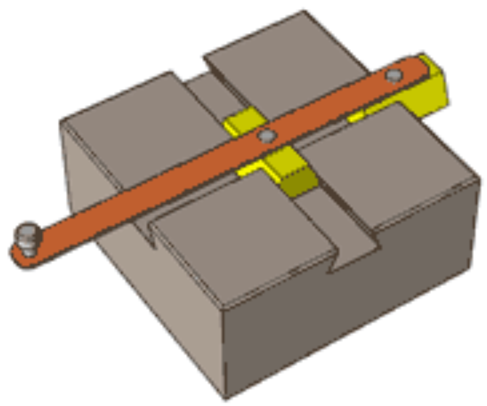

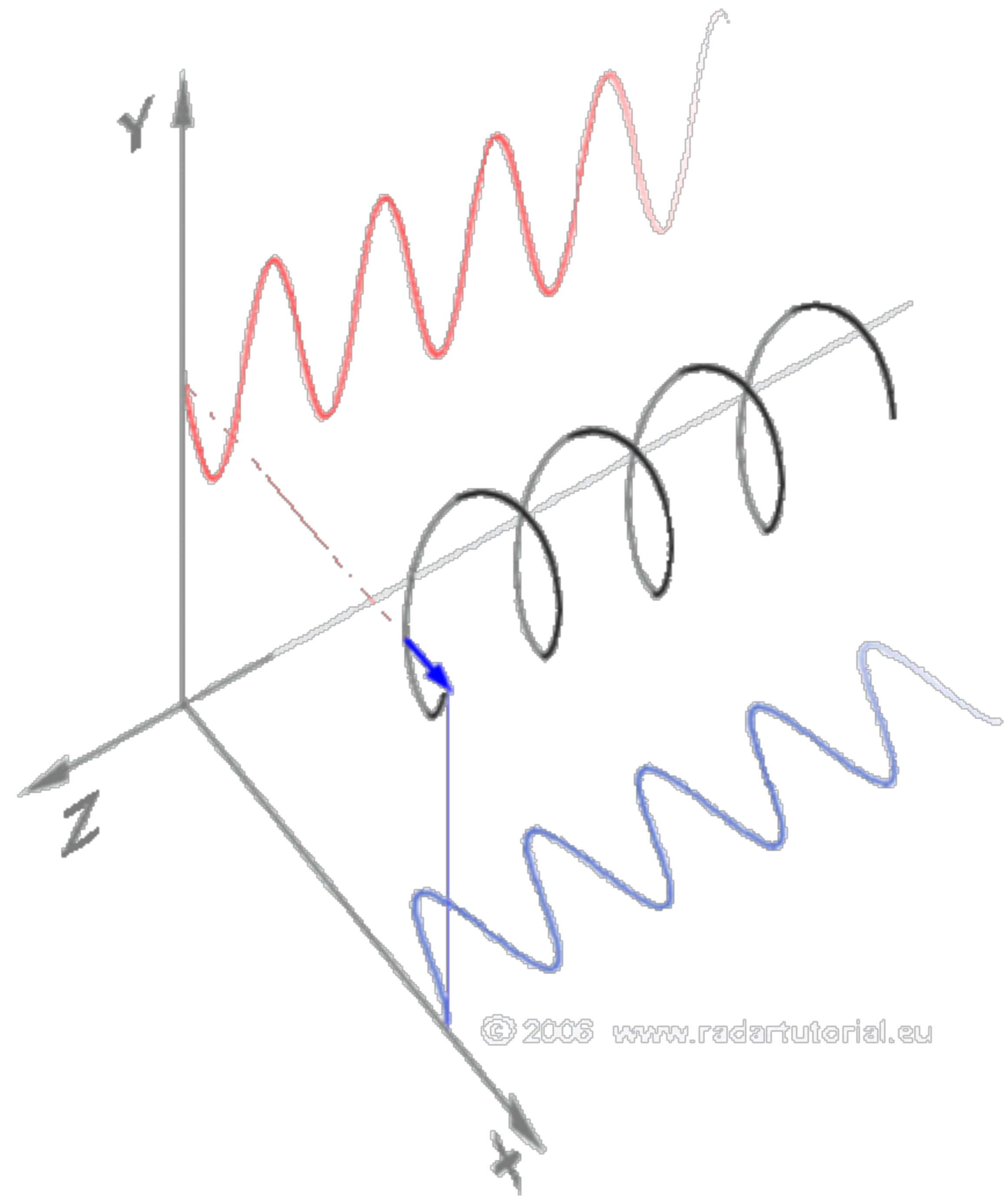

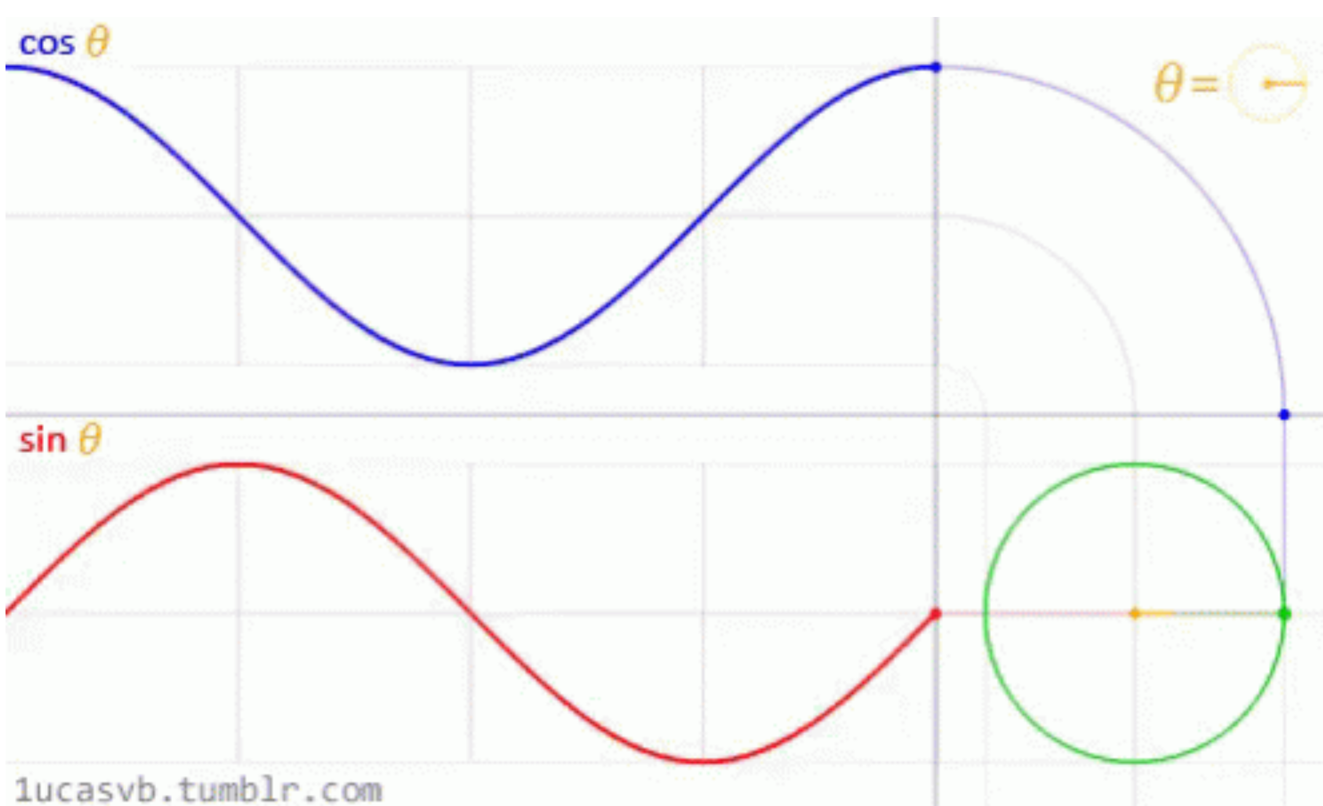

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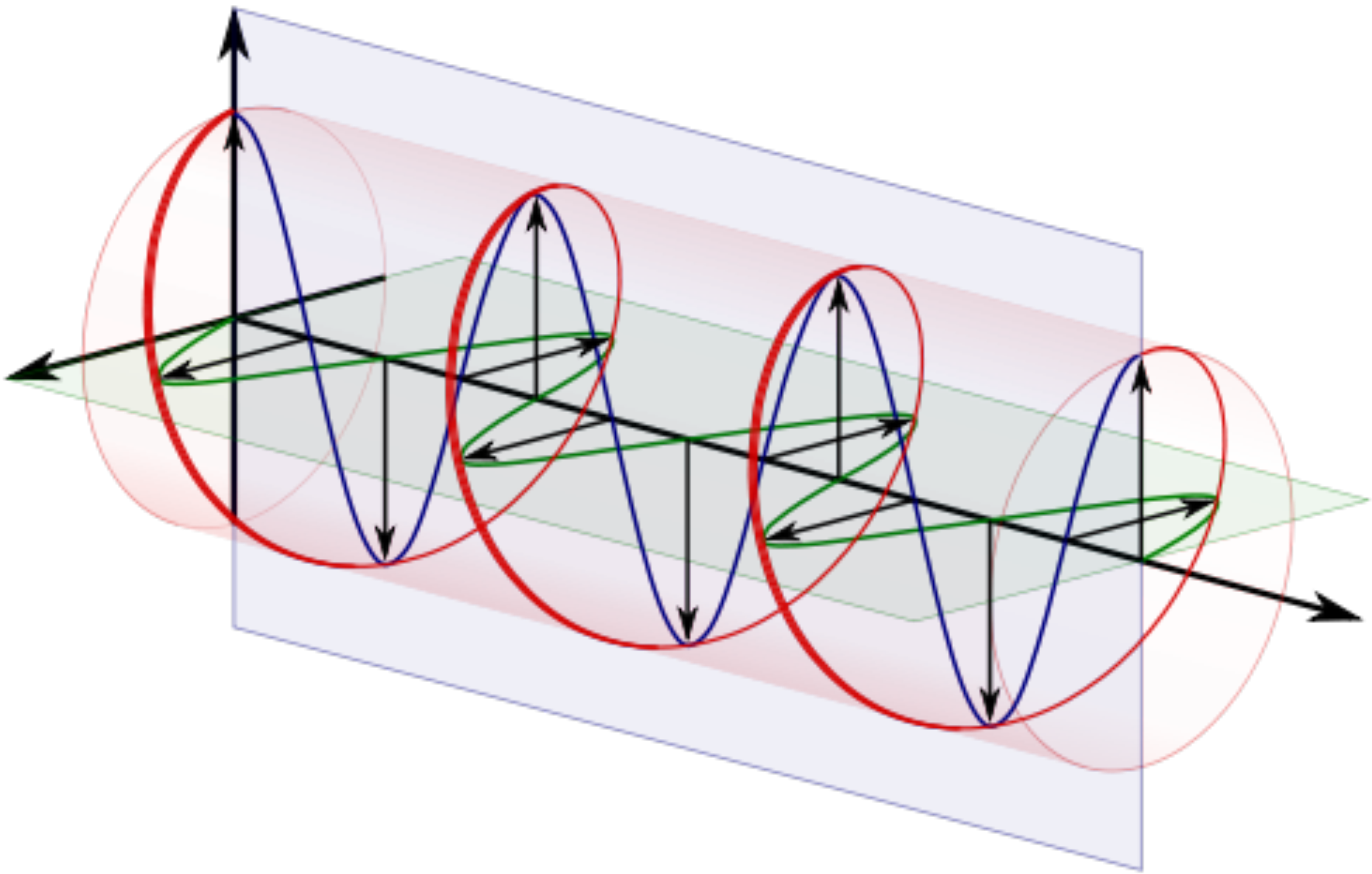
EULER'S FORMULA



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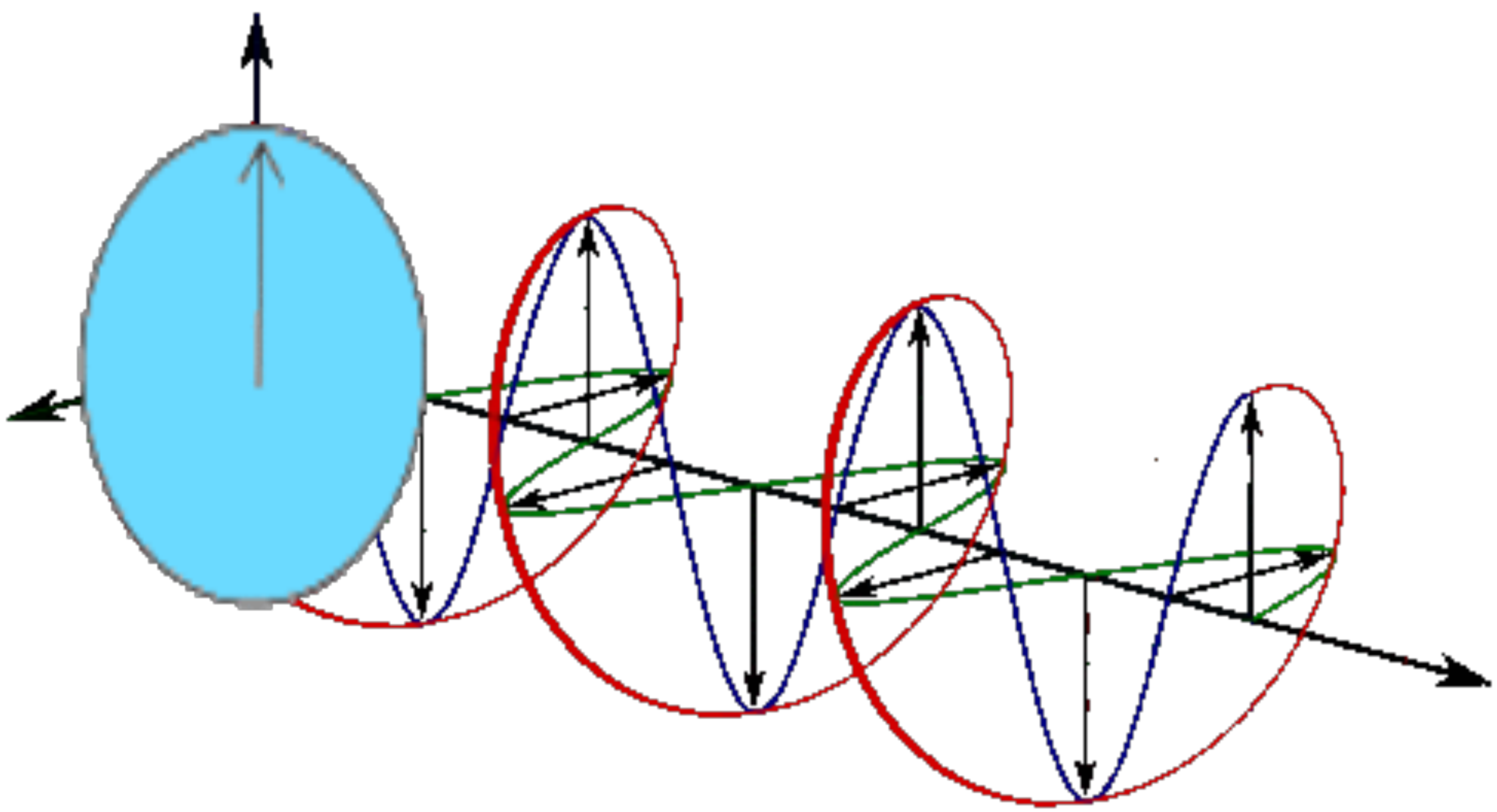

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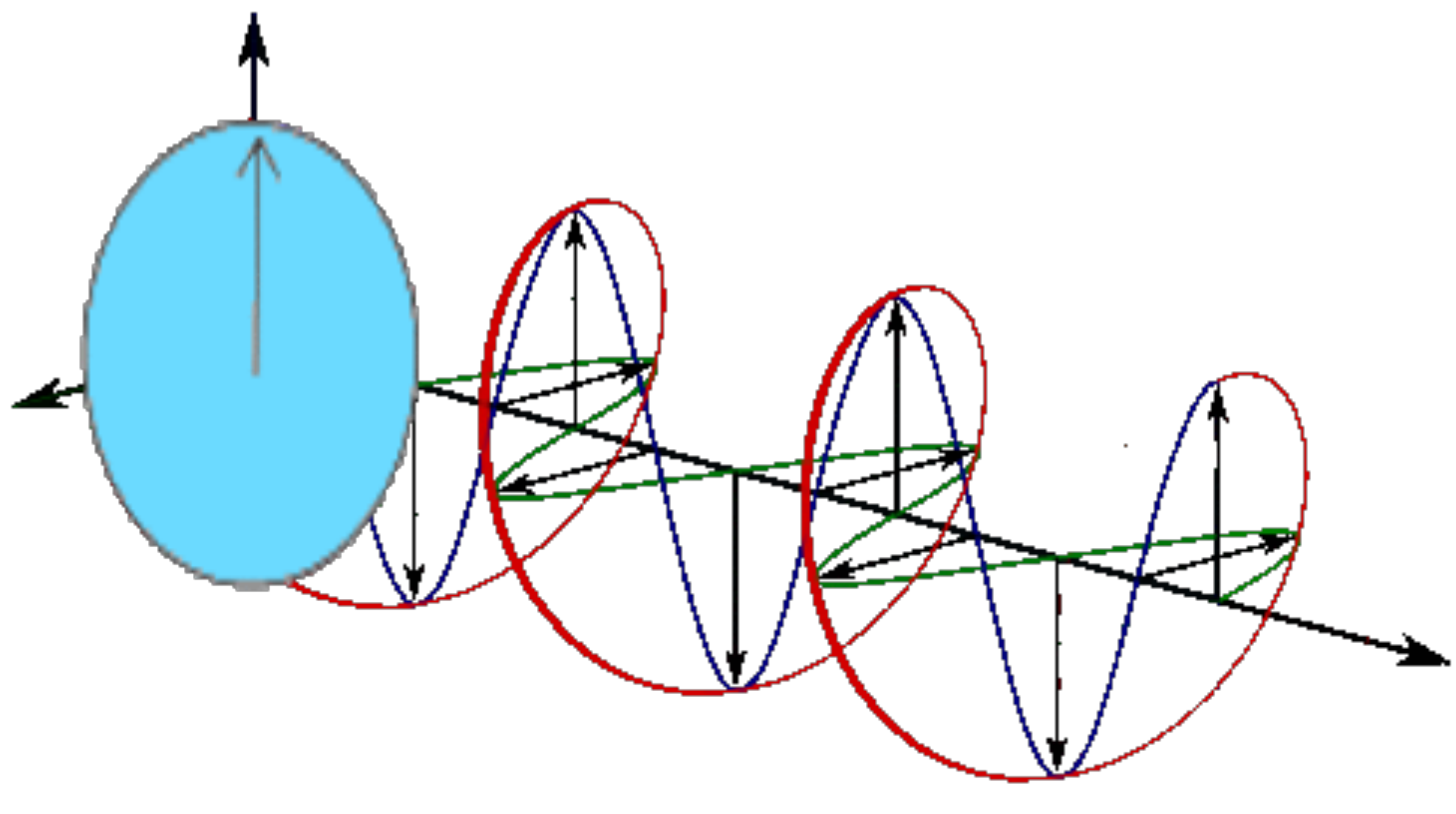

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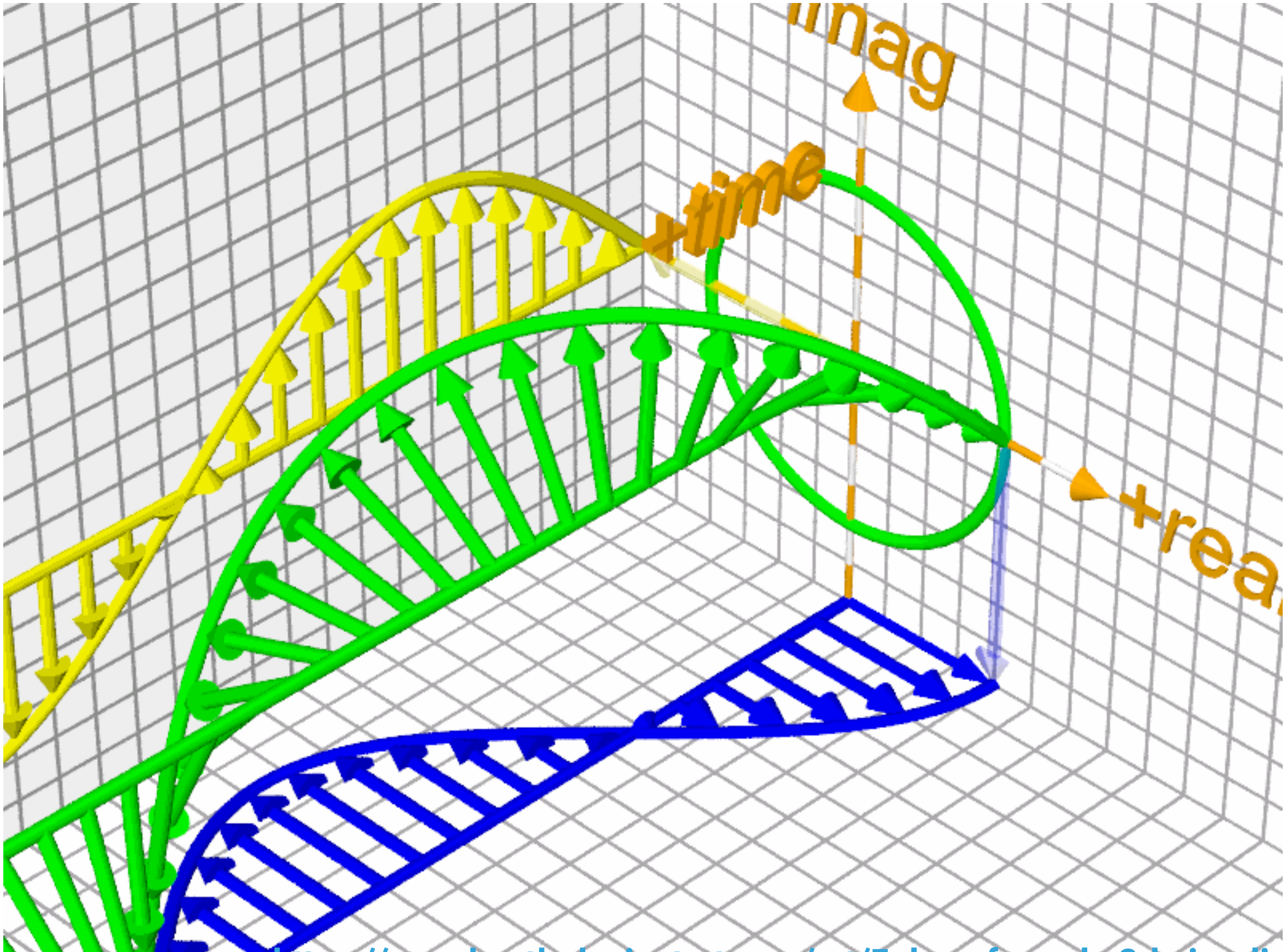

https://woodmath.deviantart.com/art/Euler-s-formula-3d-visualization-26893

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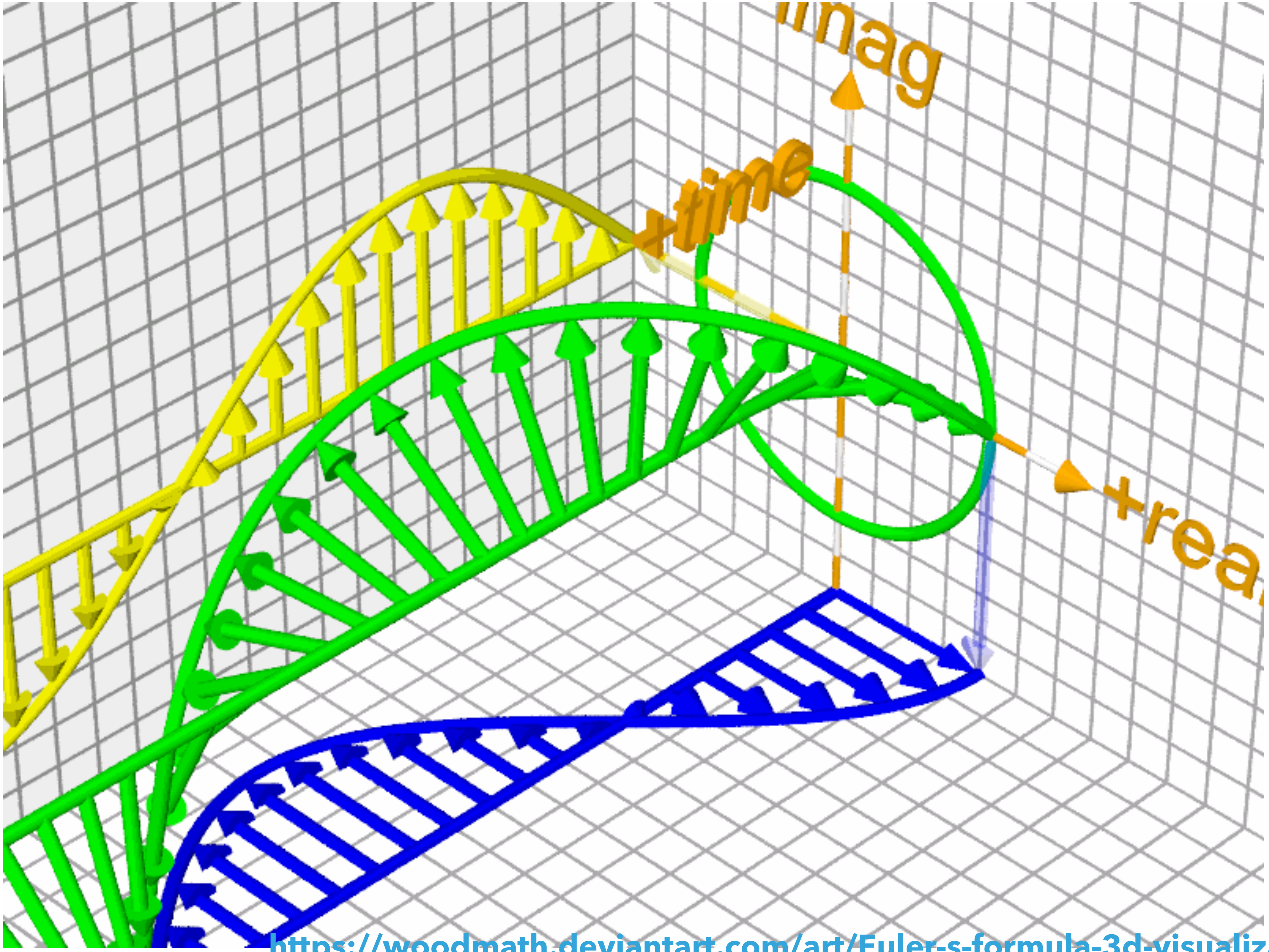
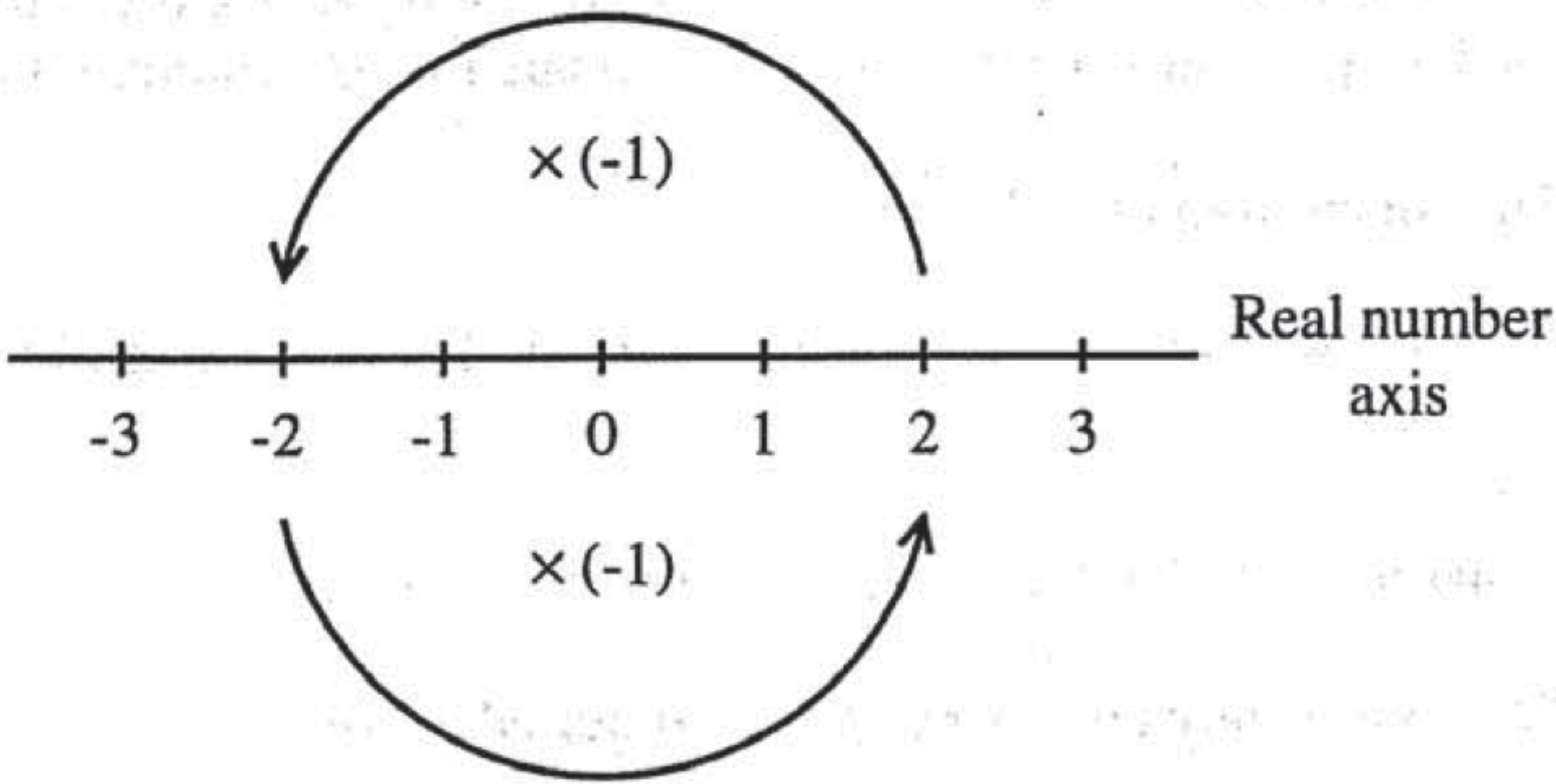


https://woodmath.deviantart.com/art/Euler-s-formula-3d-visualization-268936

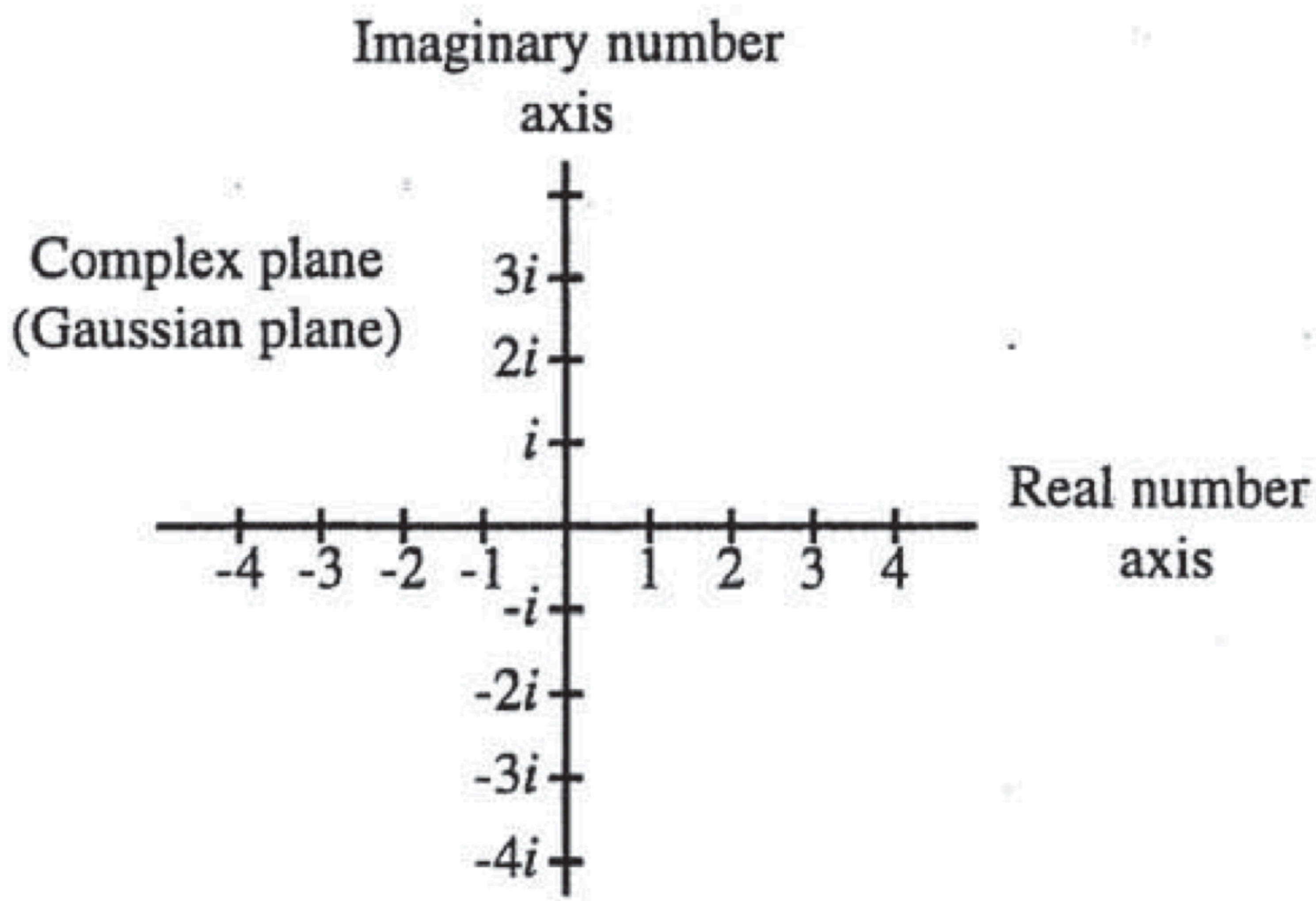
IMAGINARY NUMBERS AND ROTATION

What operation, applied twice, gets us from 1 to -1?



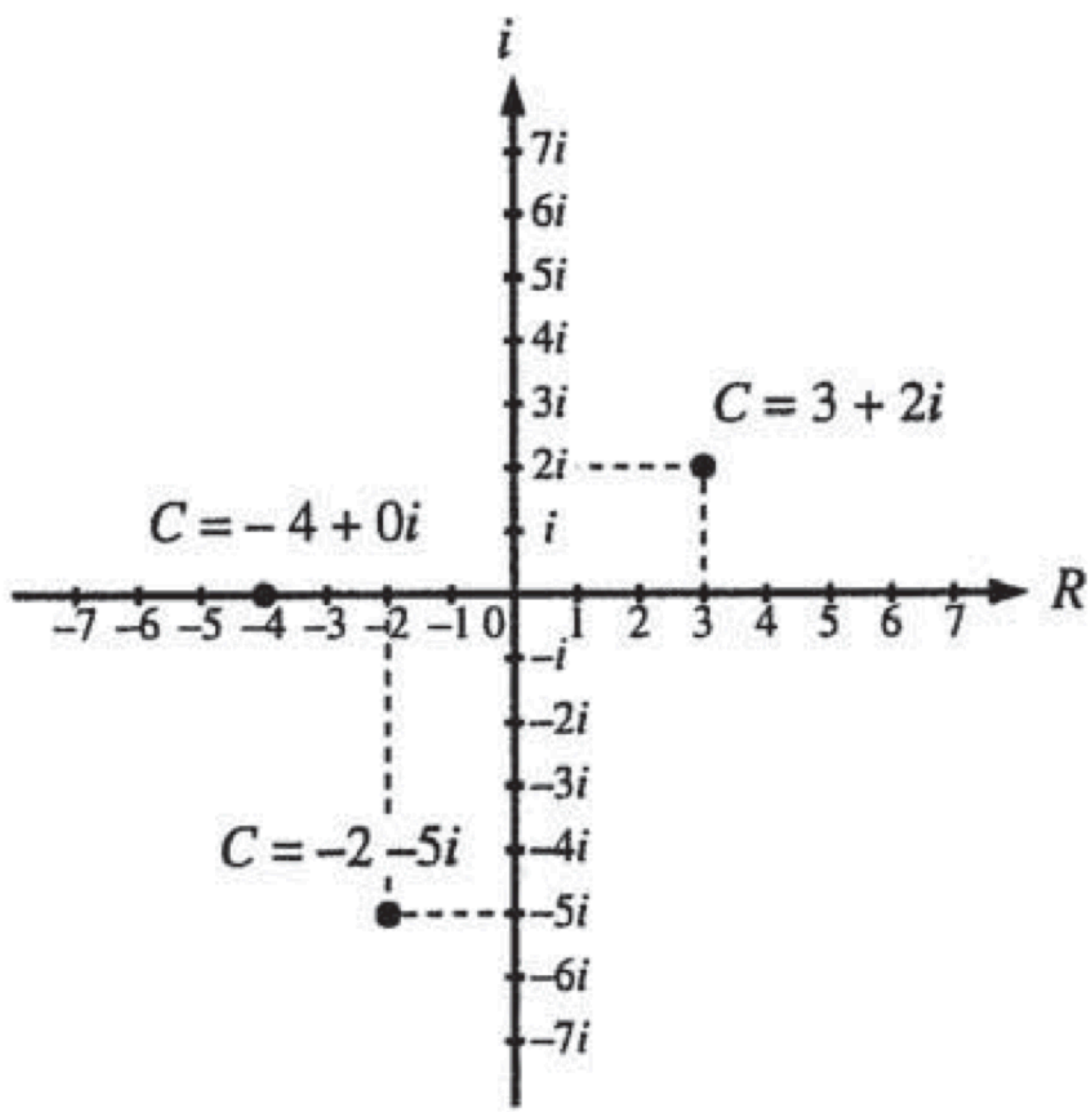
IMAGINARY NUMBERS AND ROTATION

Multiplication by *i* is a 90 degree rotation

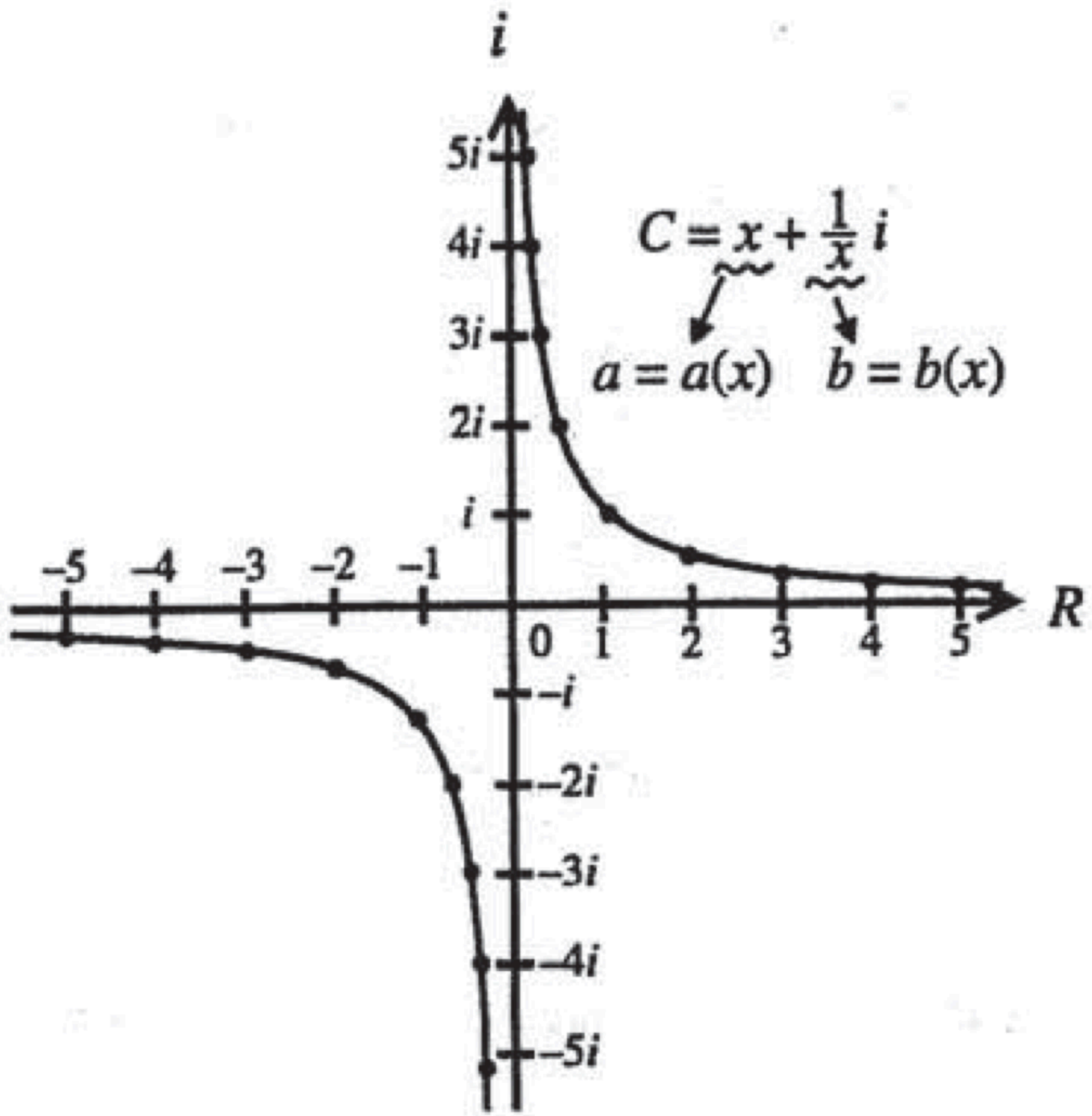


THE COMPLEX PLANE

Multiplication by *i* is a 90 degree rotation

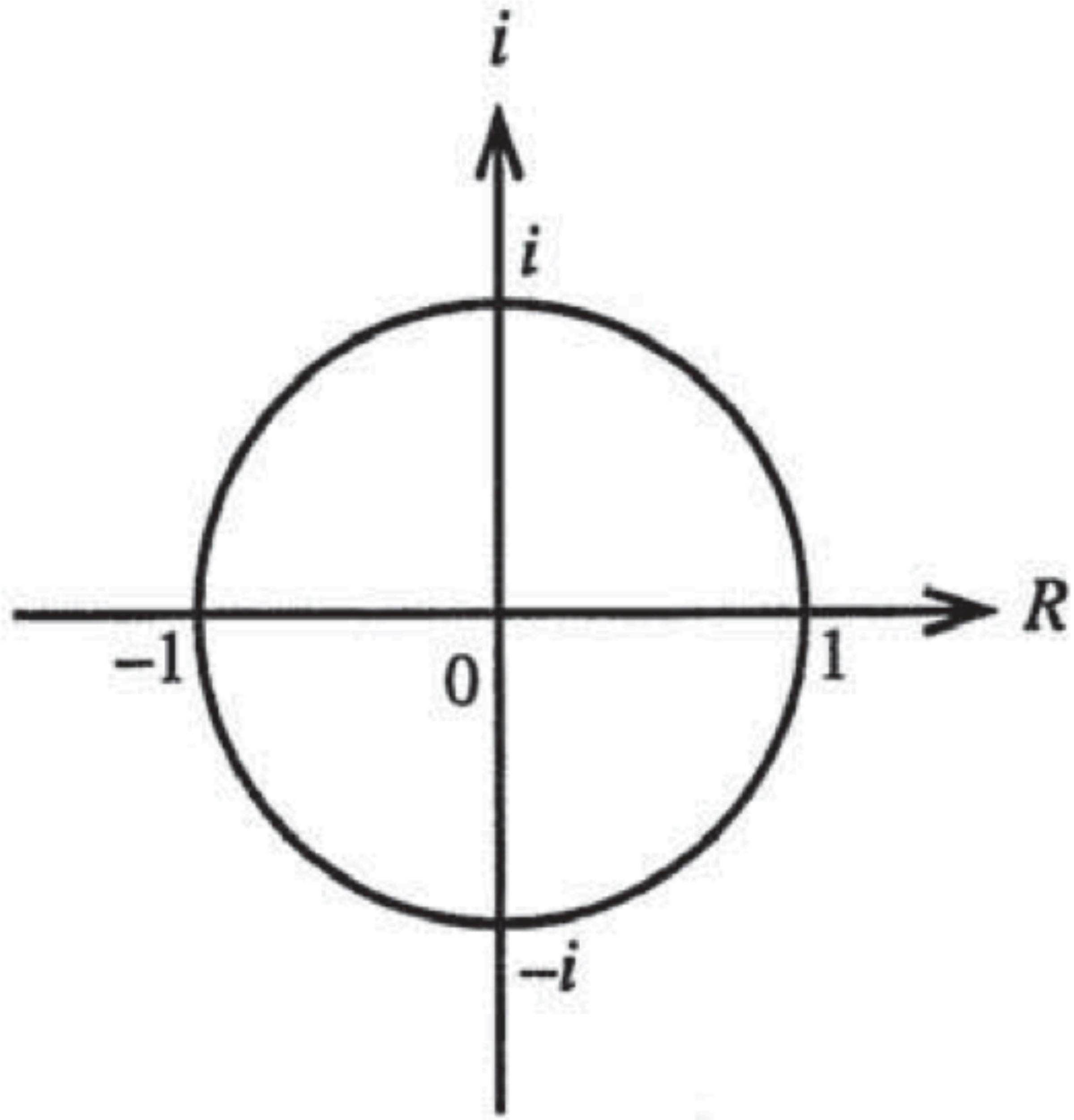


THE COMPLEX PLANE: COMPLEX FUNCTIONS



UNIT CIRCLE ON COMPLEX PLANE

$$\boxed{C(x) = a(x) + b(x)i}$$



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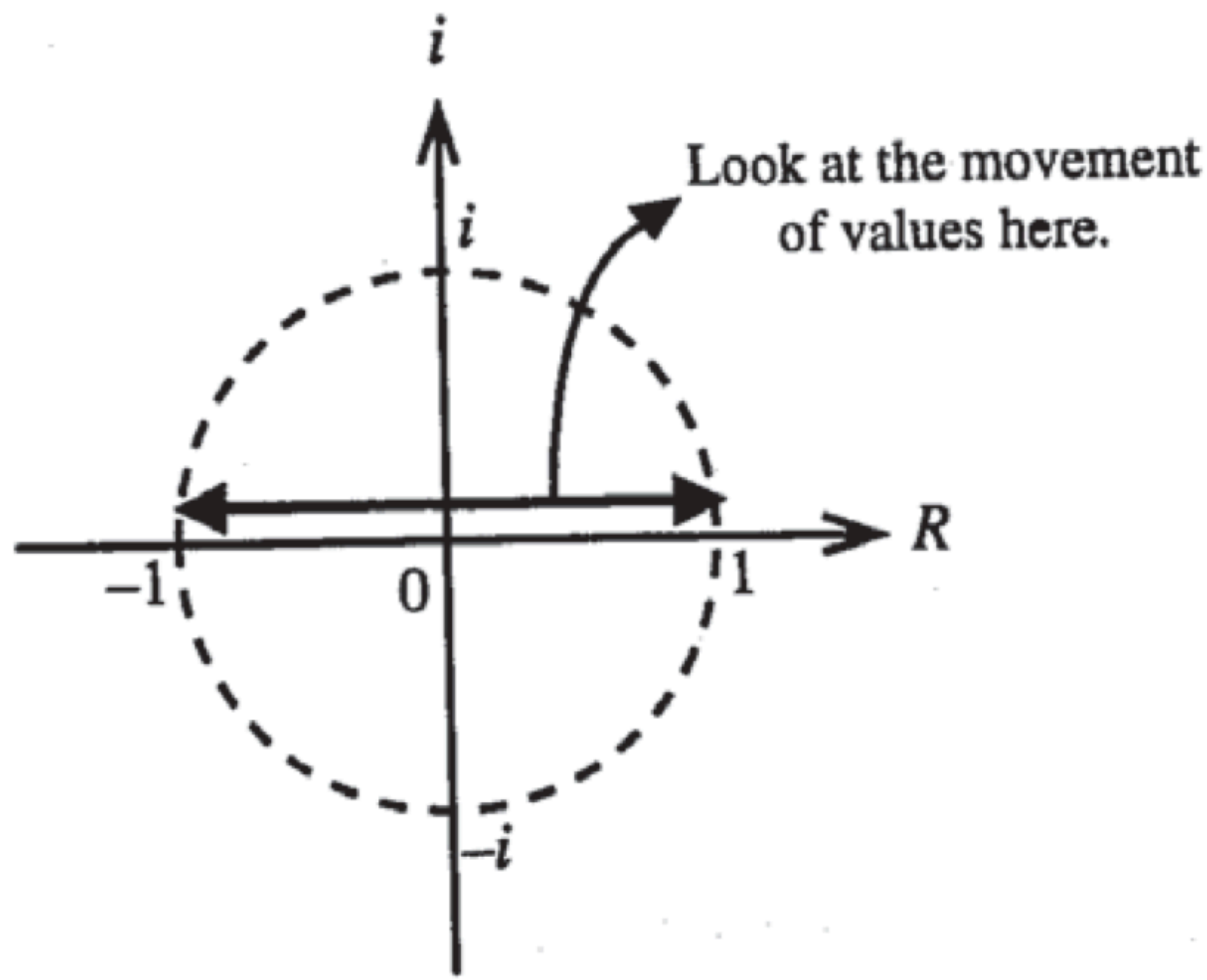
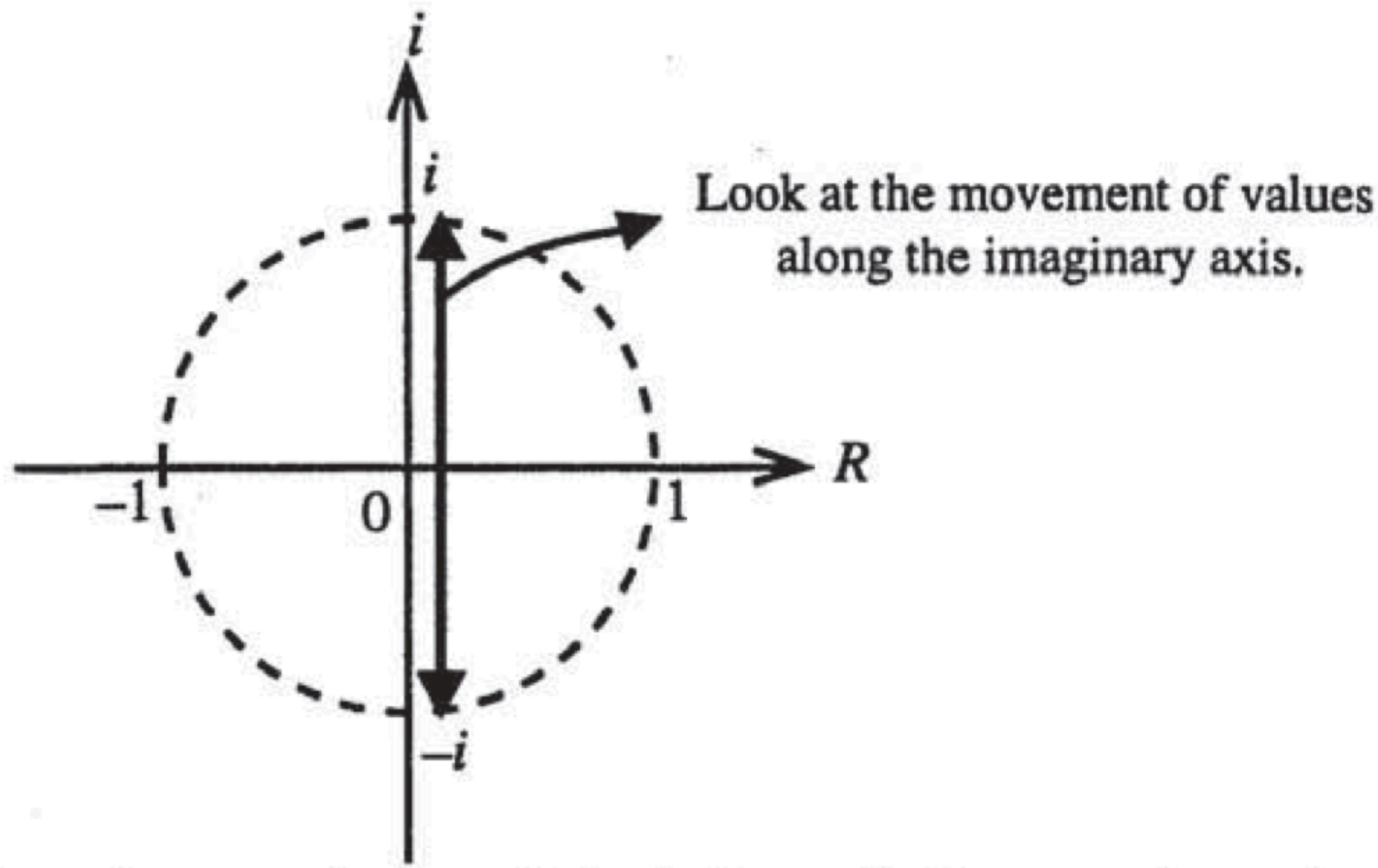
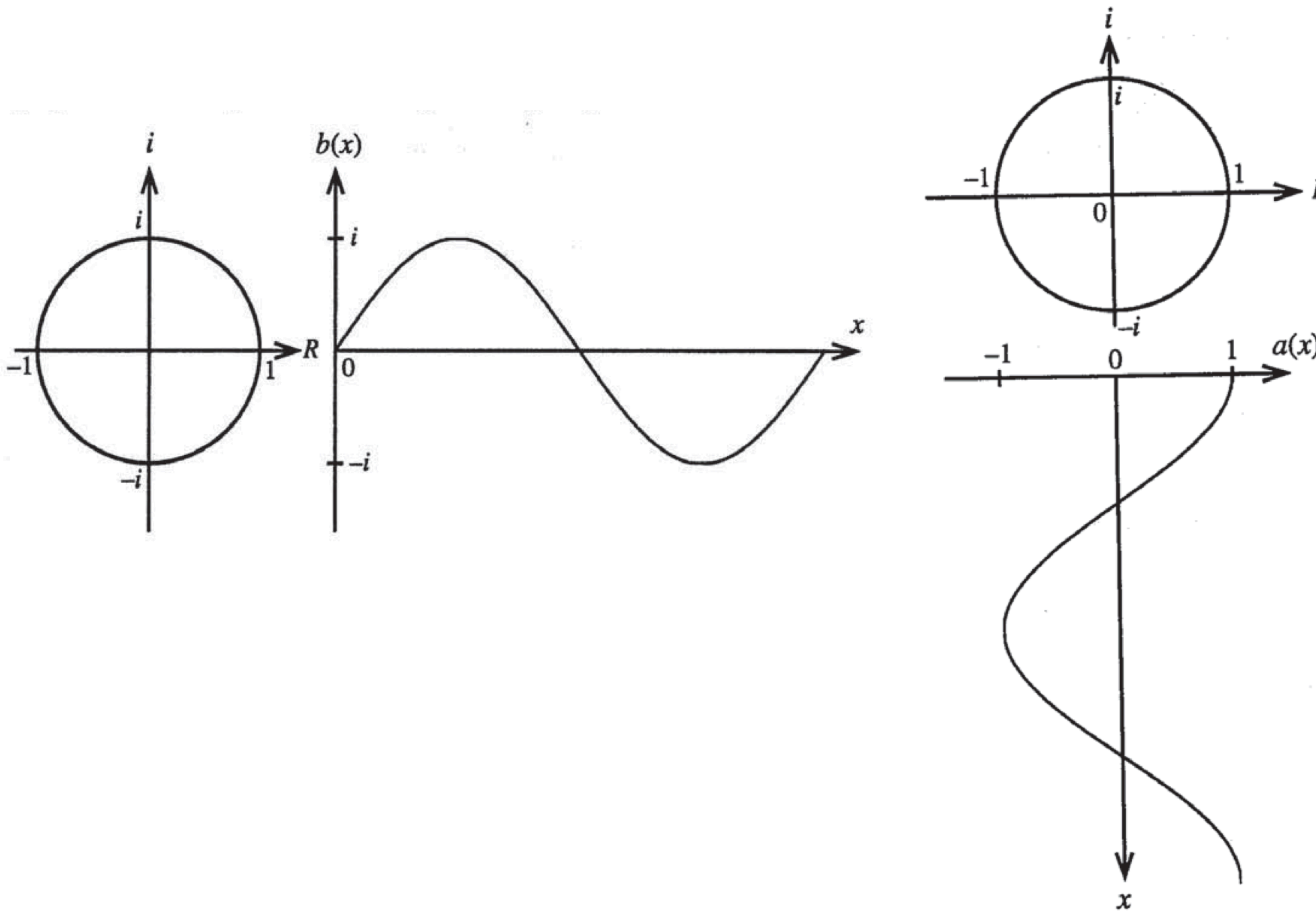


UNIT CIRCLE ON COMPLEX PLANE

$$\boxed{C(x) = a(x) + \mathbf{b(x)}i}$$

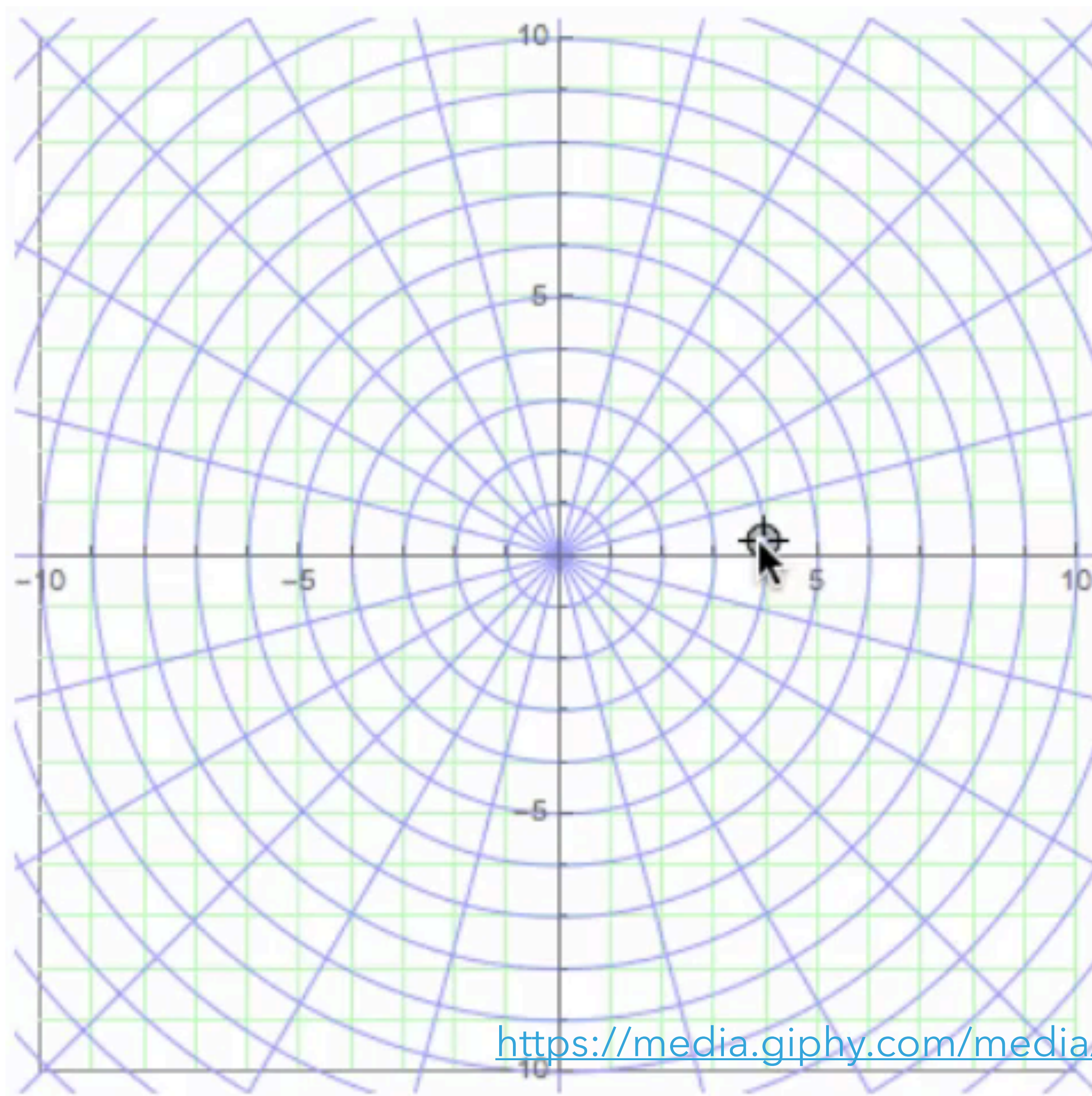


THE COMPLEX PLANE



REMINDER: POLAR COORDINATES

Polar: point $P = (r, \theta)$



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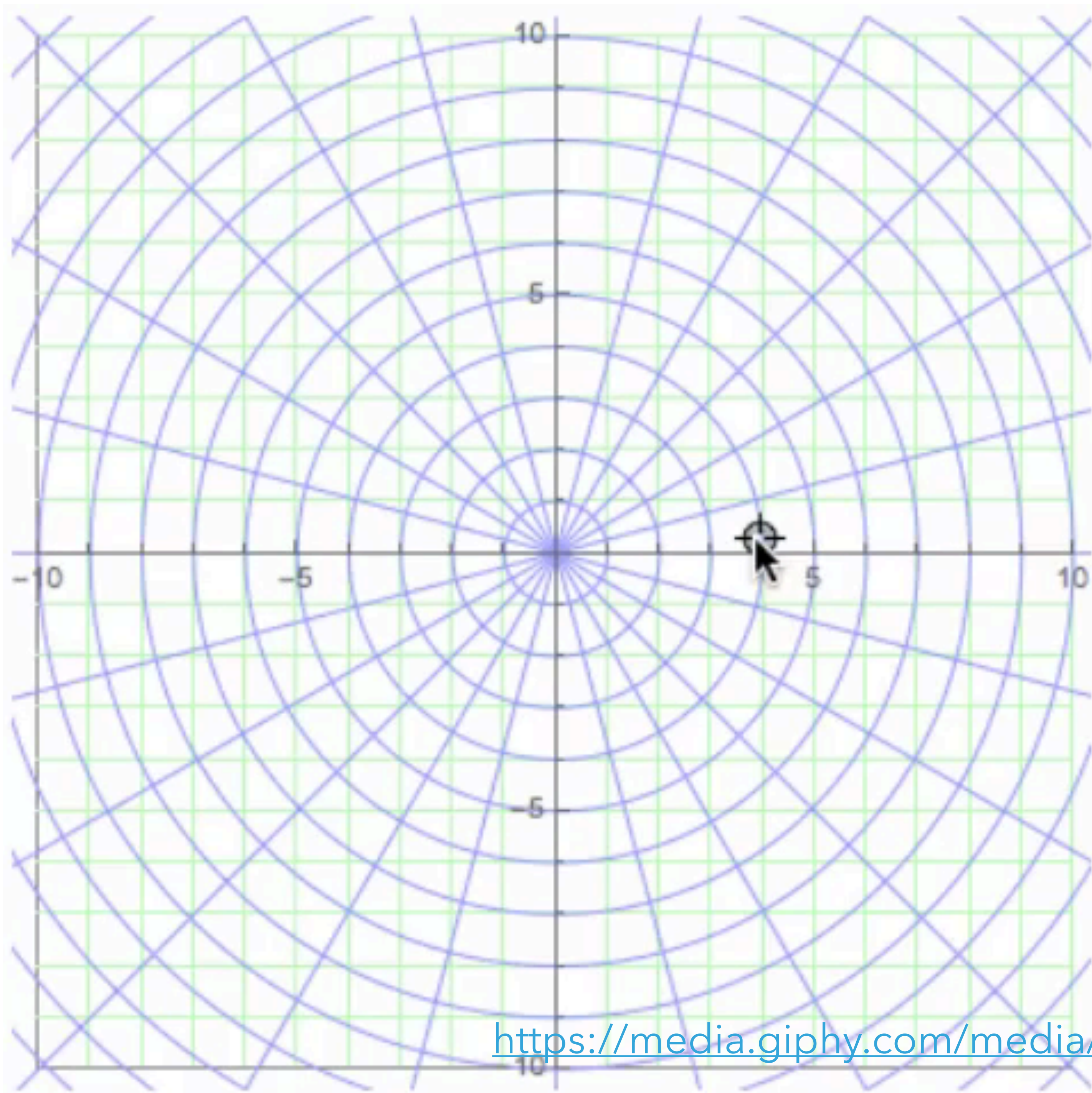
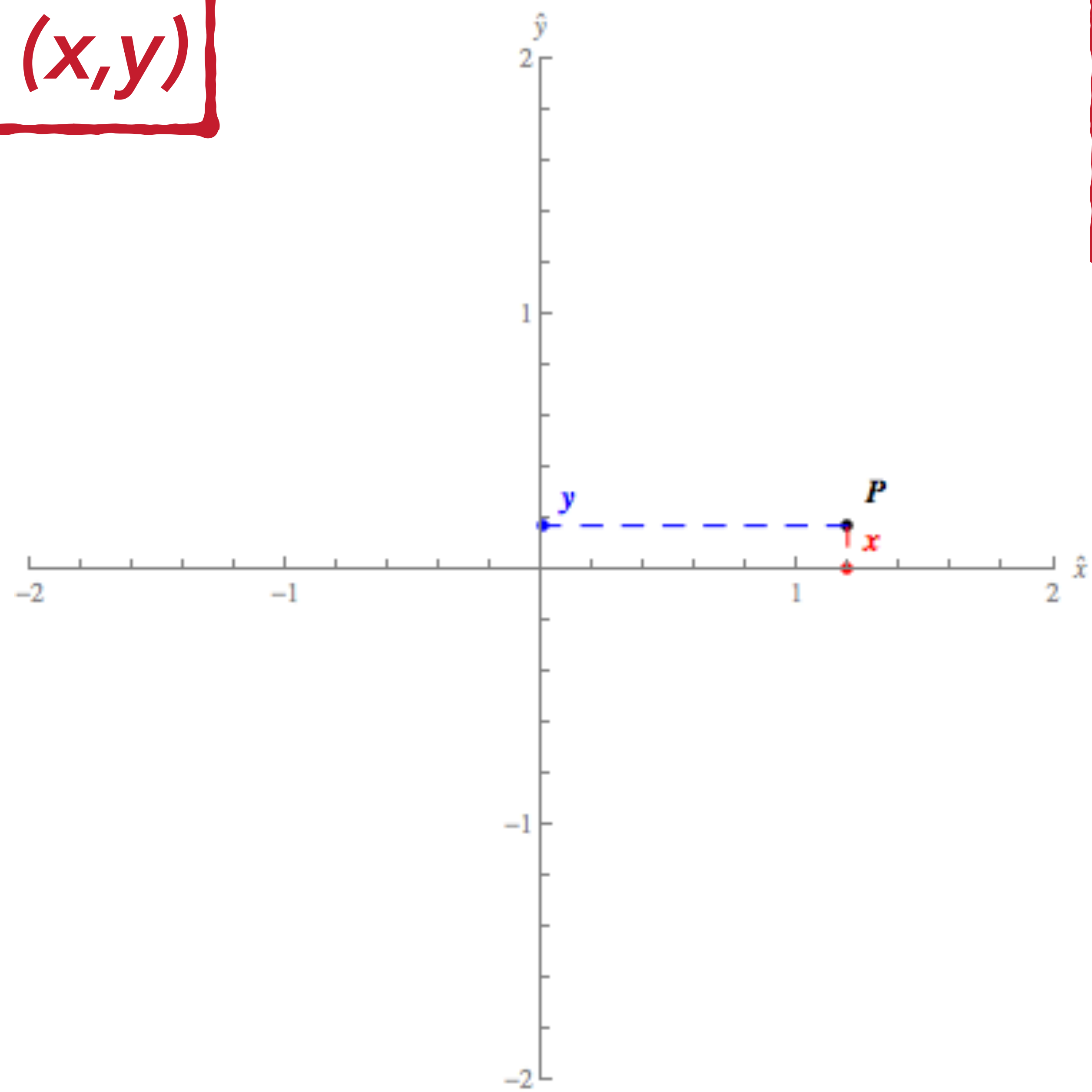


https://media.giphy.com/media/sGjQtk2jLDUdy/giphy.gif

RECTANGULAR ↔ POLAR COORDINATES

P = (x, y)
Traversal via *(x,y)*



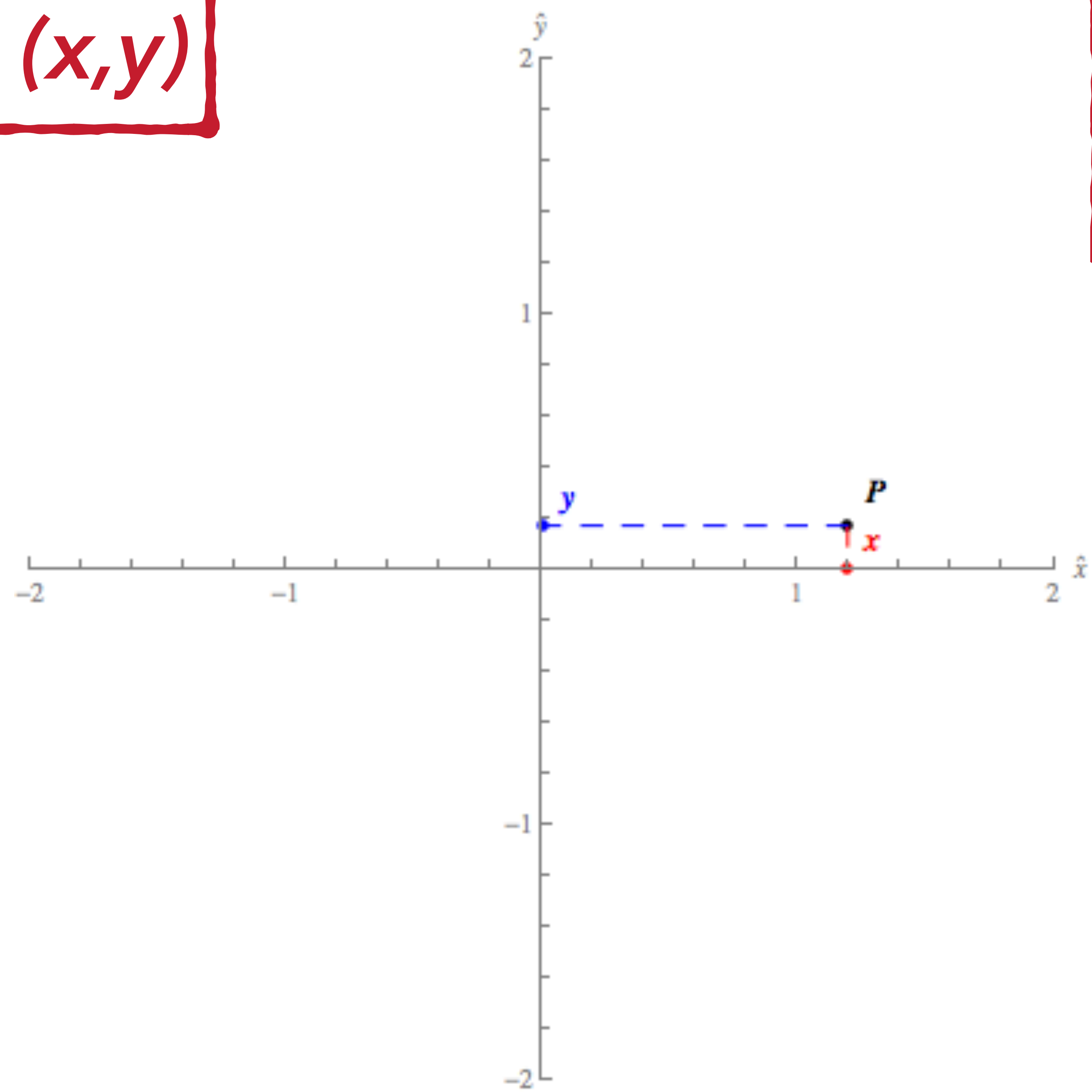
P = (r, θ)
Stretch by *r*,
rotation by *θ*

https://media.giphy.com/media/NVqZEZT4WBmul/giphy.gif

RECTANGULAR ↔ POLAR COORDINATES

$P = (x, y)$
Traversal via (x,y)

$P = (r, \theta)$
Stretch by r,
rotation by θ



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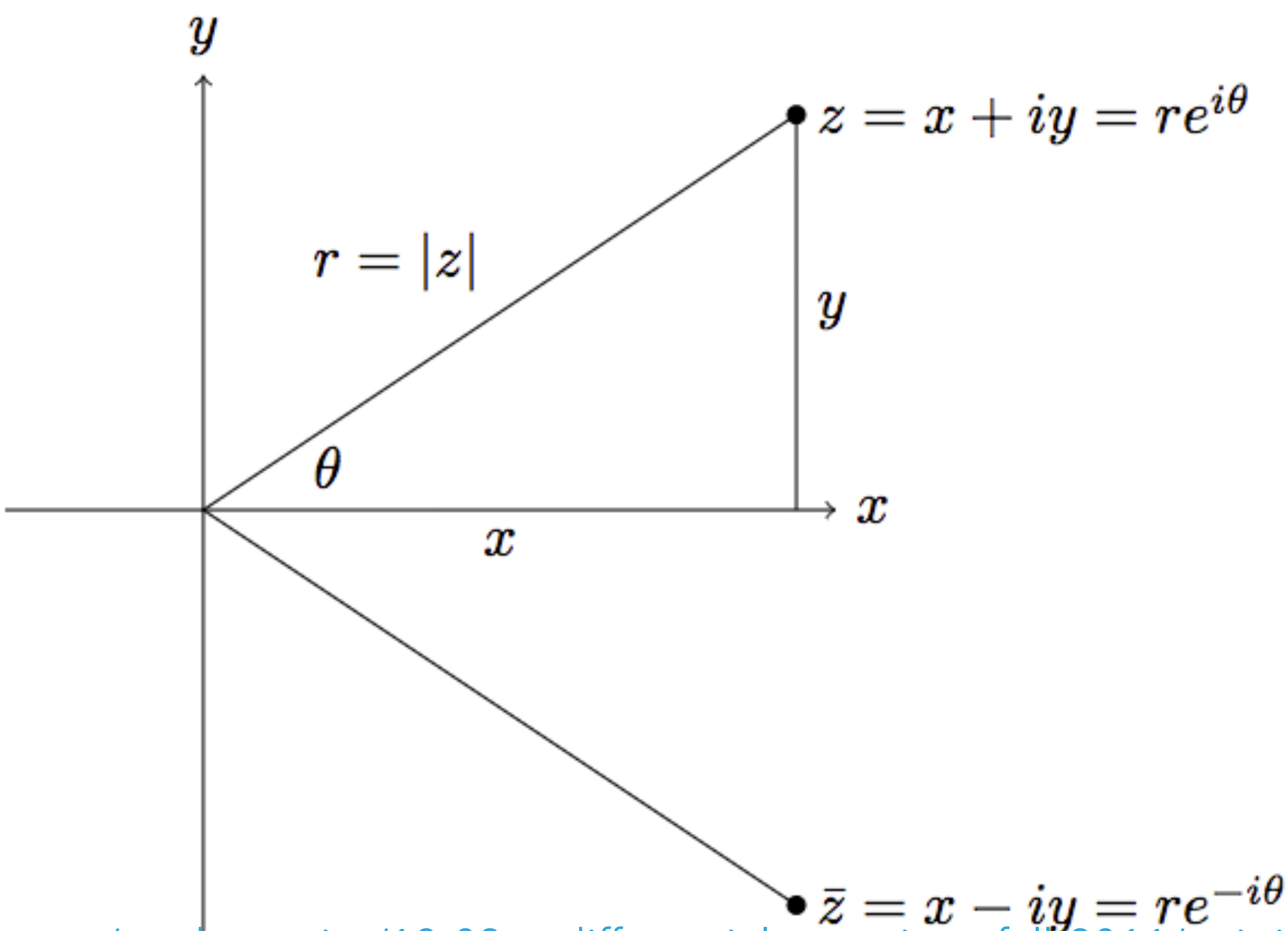
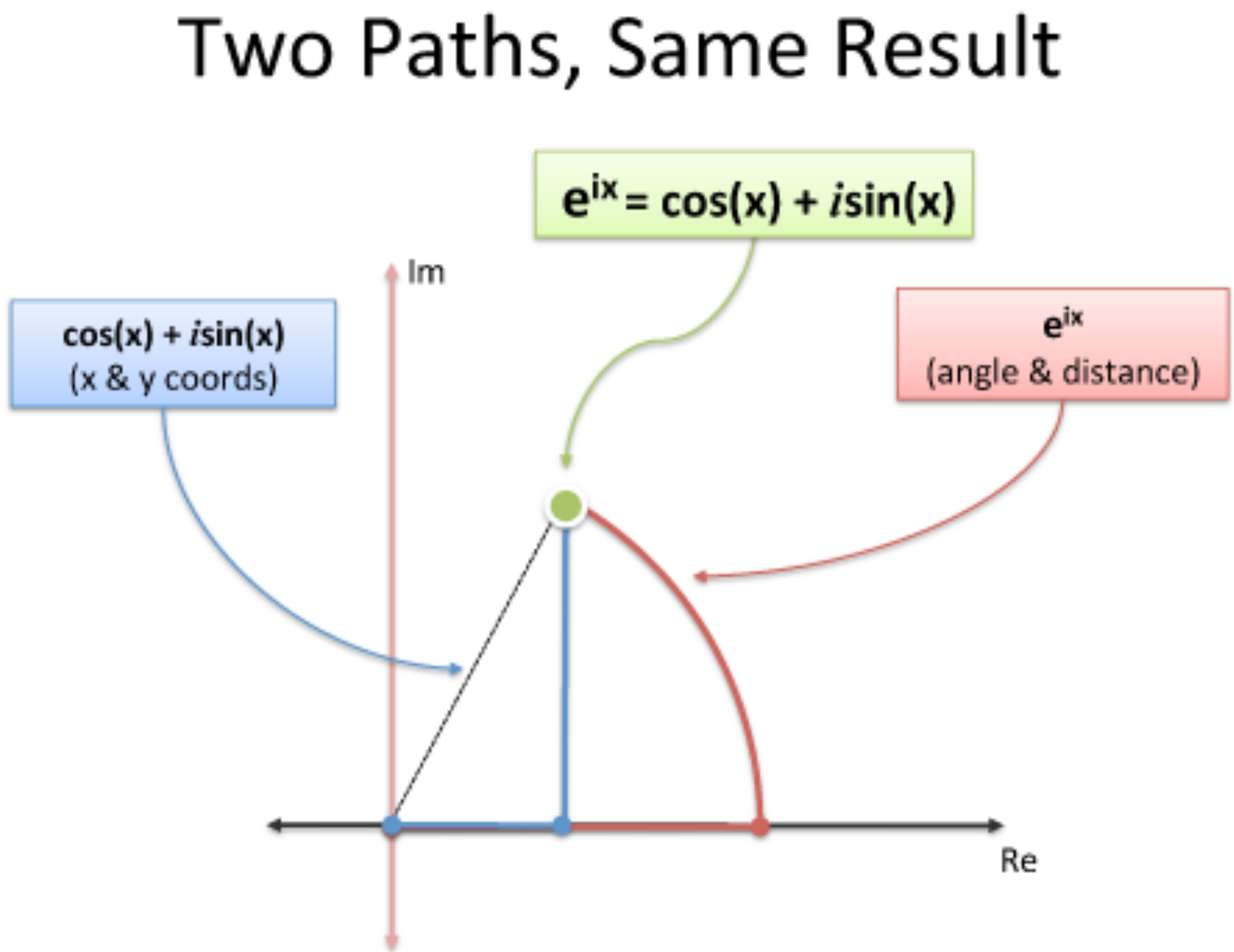
GEOMETRIC EQUIVALENCE



https://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-i-first-order-differential-equations/complex-arithmetic-and-exponentials/MIT18_03SCF11_s6_5text.pdf

EULER'S FORMULA



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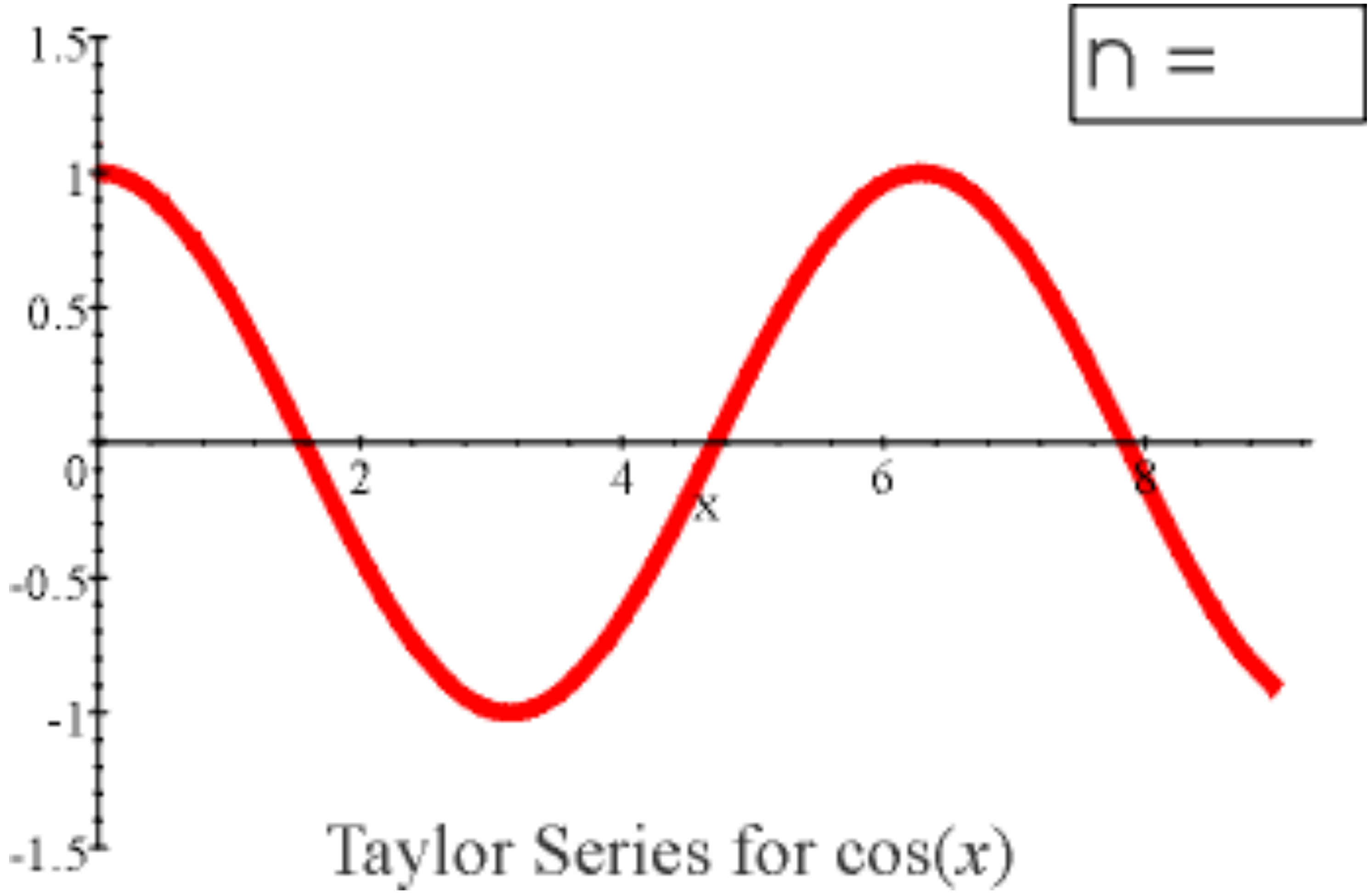
TAYLOR SERIES FOR COS(X)



http://www.math.tamu.edu/~dallen/cos.gif

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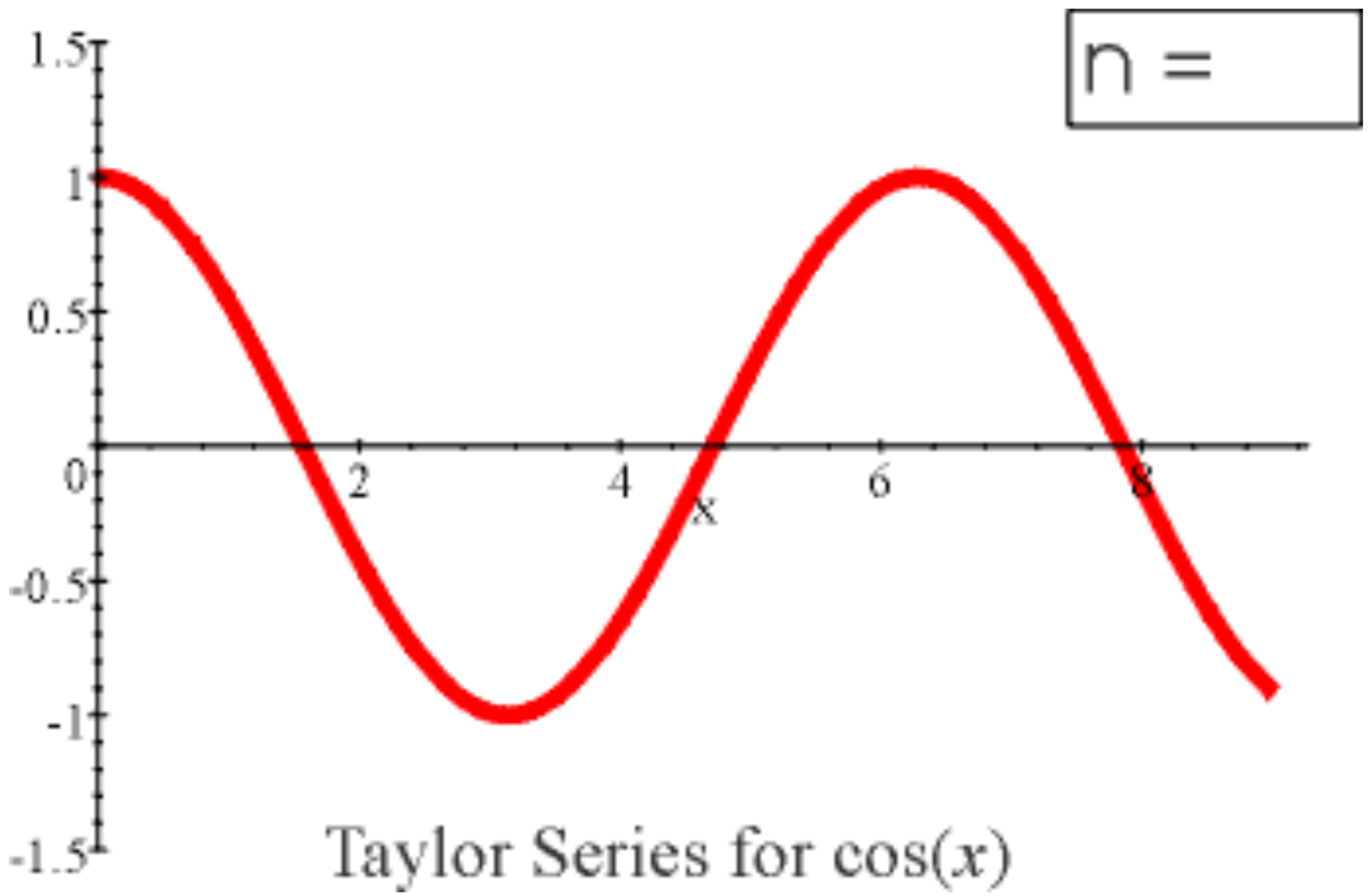

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EULER'S FORMULA ALGEBRAICALLY

$$\cos(x) = 1 + 0x - \frac{1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 + 0x^5 - \frac{1}{6!}x^6 + 0x^7 + \frac{1}{8!}x^8 + \ldots$$

$$i\sin(x) = 0 + ix + 0x^2 - i\frac{1}{3!}x^3 + 0x^4 + i\frac{1}{5!}x^5 + 0x^6 - i\frac{1}{7!}x^7 + 0x^8 + \ldots$$

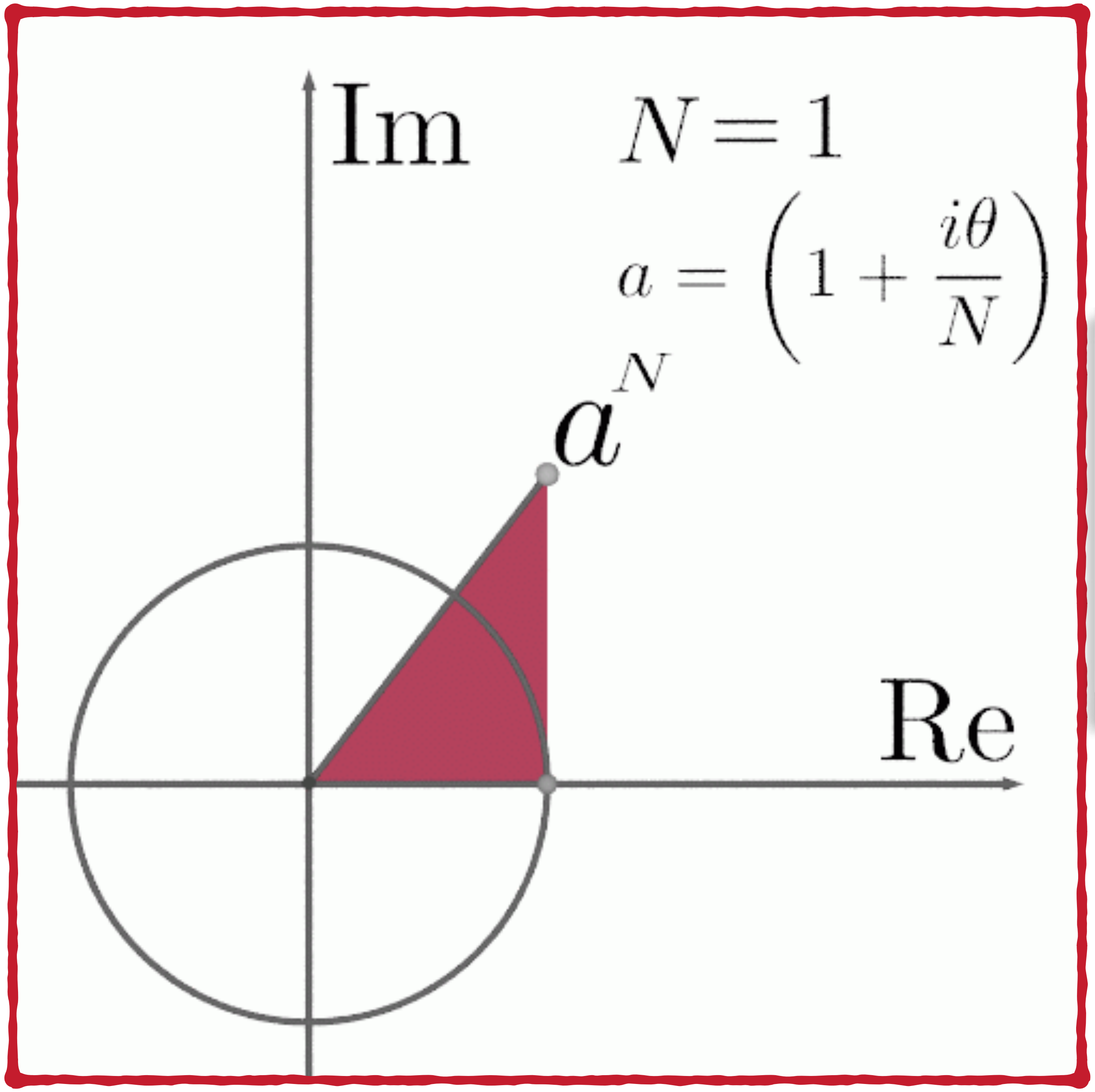
$$e^{ix} = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \ldots$$

REMINDER: WHAT IS e ?

$$\boxed{e^z = \lim_{N \to \infty} \left(1 + \frac{z}{N}\right)^N}$$

- **e = 2.718281828 . . .**
- **Special property:**
  - When you differentiate or integrate e^x you get back e^x

ANOTHER LIMIT EXPLANATION GRAPHICALLY



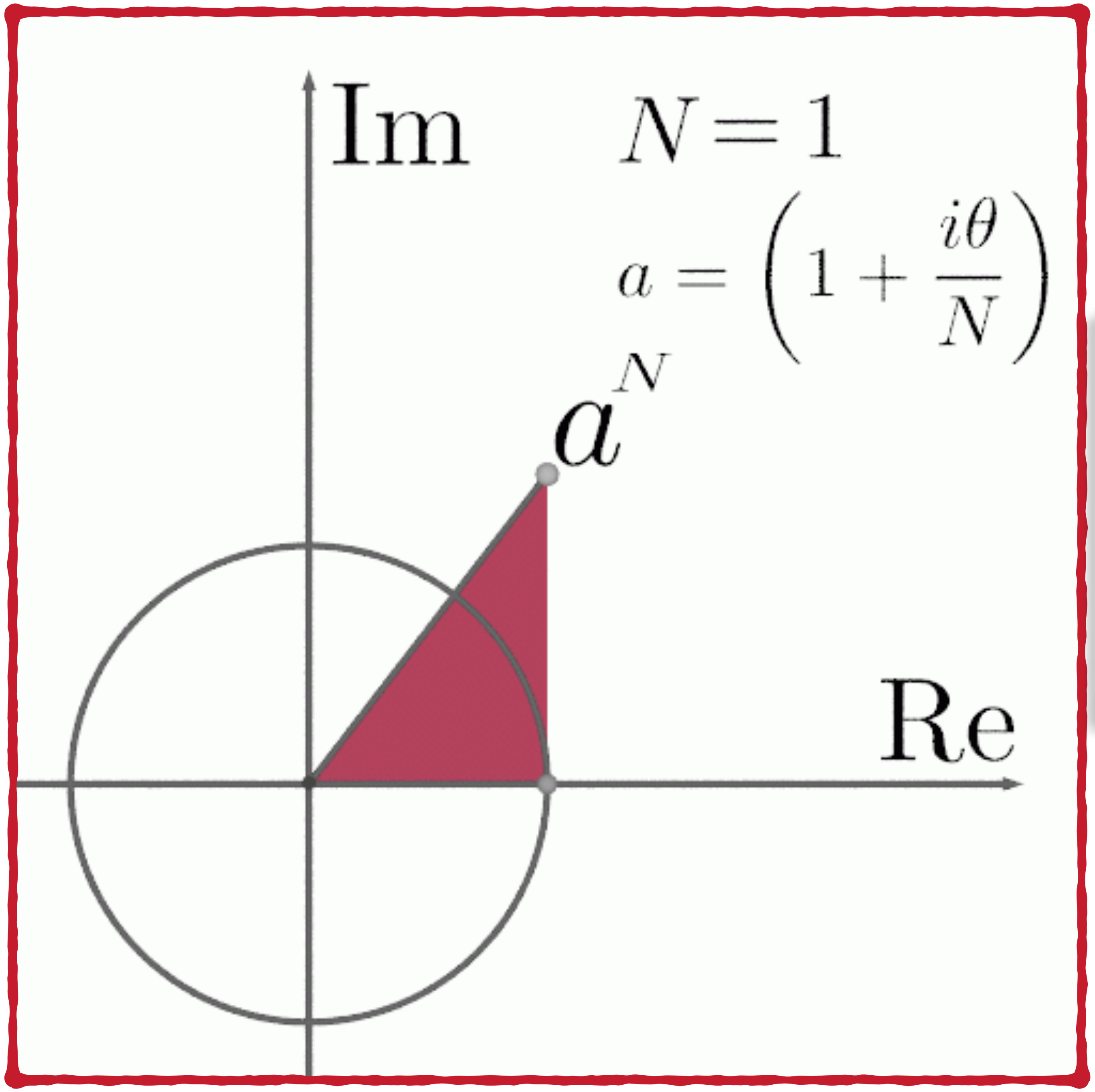
$$e^z = \lim_{N\to\infty}\left(1 + \frac{z}{N}\right)^N$$

$$e^{i\theta} = \lim_{N\to\infty}\left(1 + \frac{i\theta}{N}\right)^N = \lim_{N\to\infty} a^N = \cos\theta + i\sin\theta$$

$$e^{i\tau} = 1$$

http://hyrodium.tumblr.com/post/106436002954/euler2

ANOTHER LIMIT EXPLANATION GRAPHICALLY



$$e^z = \lim_{N\to\infty} \left(1 + \frac{z}{N}\right)^N$$

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$$e^{i\tau} = 1$$

http://hyrodium.tumblr.com/post/106436002954/euler2

COMPLEX NUMBER REPRESENTATIONS OF SINUSOIDS

Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$

In-class Exercise:

Using Euler's formula, can you re-express cos(x) and sin(x) in terms of e, i, and x?

COMPLEX REP. OF FOURIER SERIES

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi int}$$

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi int} f(t)\,dt$$

COMPLEX REP. OF FOURIER SERIES

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi i n t}$$

$$C_n = \frac{1}{T} \int_0^T e^{-2\pi i n t} f(t)\, dt$$

See Who is Fourier Chs. 10,11
Osgood notes Sec 1.6.1

TOWARDS
THE FOURIER TRANSFORM:
KEY IDEAS

TOWARDS THE FOURIER TRANSFORM: KEY IDEAS

See Who is Fourier Ch. 12

FOURIER TRANSFORM "FAMILY"

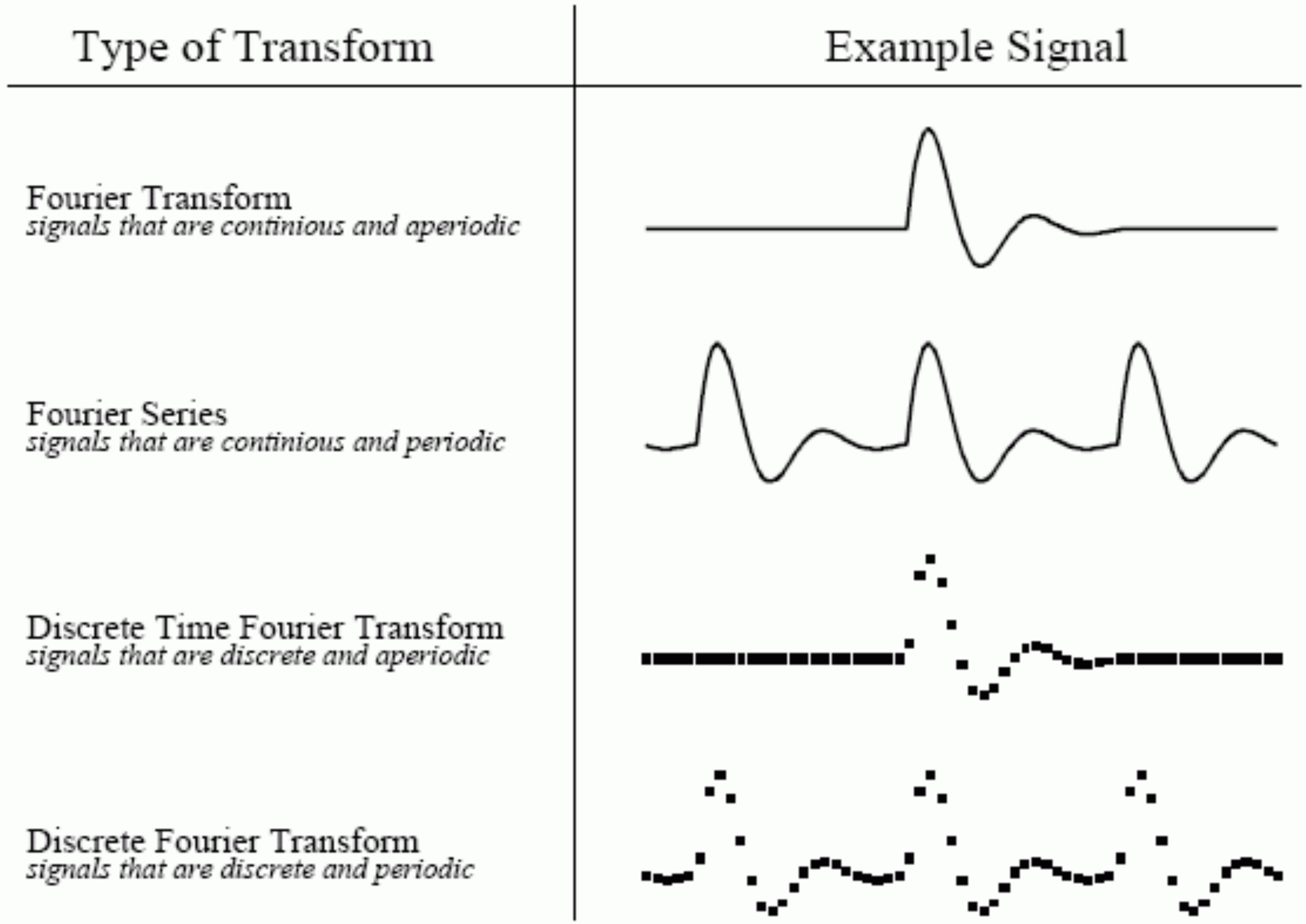
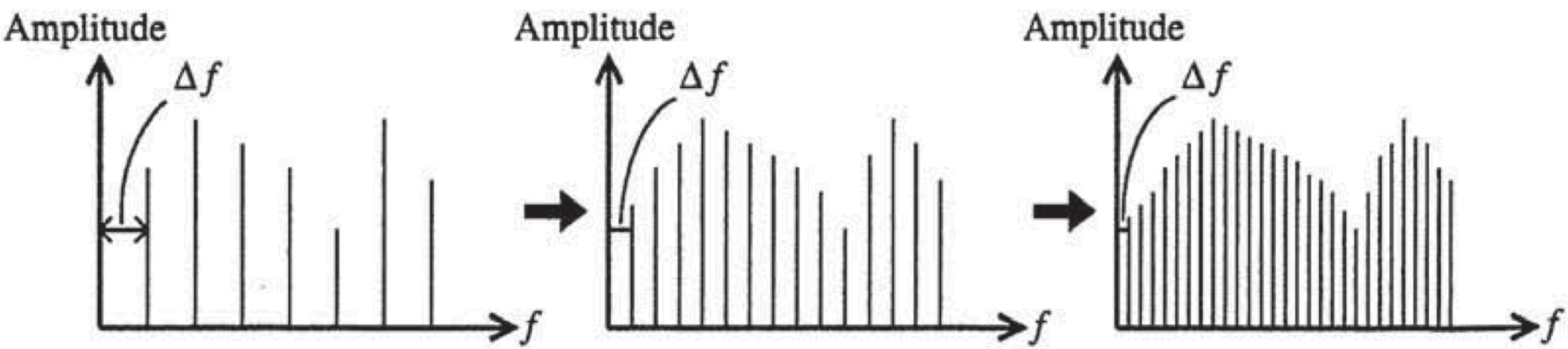


FIGURE 8-2
Illustration of the four Fourier transforms. A signal may be continuous or discrete, and it may be periodic or aperiodic. Together these define four possible combinations, each having its own version of the Fourier transform. The names are not well organized; simply memorize them.

http://www.dspguide.com/ch8/1.htm

APERIODIC FUNCTIONS: "INFINITE" PERIOD

1. frequency resolution $\Delta f = \frac{1}{T}$
2. As $T \to \infty$, $\Delta f \to 0$



WE START WITH THE FOURIER SERIES…

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n t/T}$$

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi i n t/T} f(t)\,dt$$

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Here we generalize to arbitrary (finite) period T

FOURIER TRANSFORM DERIVATION II

So we can rewrite this:

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi int/T} f(t)dt$$

as (with a shift to the left by T/2):

$$C_n = \Delta f \int_{-T/2}^{T/2} e^{-2\pi i f_n t} f(t)dt$$

FOURIER TRANSFORM DERIVATION III

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n t/T}$$

$$C_n = \Delta f \int_{-T/2}^{T/2} e^{-2\pi i f_n t} f(t) dt$$

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} \Delta f \int_{-T/2}^{T/2} e^{-2\pi i f_n t} g(t) dt e^{2\pi i f_n t}}$$

FOURIER TRANSFORM DERIVATION III

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n t/T}$$

Rewrite C_n with our new formula and rename f(t) as g(t)

$$C_n = \Delta f \int_{-T/2}^{T/2} e^{-2\pi i f_n t} f(t) dt$$

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} \Delta f \int_{-T/2}^{T/2} e^{-2\pi i f_n t} g(t) dt e^{2\pi i f_n t}}$$

FOURIER TRANSFORM DERIVATION IV

Finally, take the limit as $T \to \infty$:

$$g(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \int_{-T/2}^{T/2} e^{-2\pi i f_n t} g(t) dt e^{2\pi i f_n t} \Delta f$$

FOURIER TRANSFORM DERIVATION V

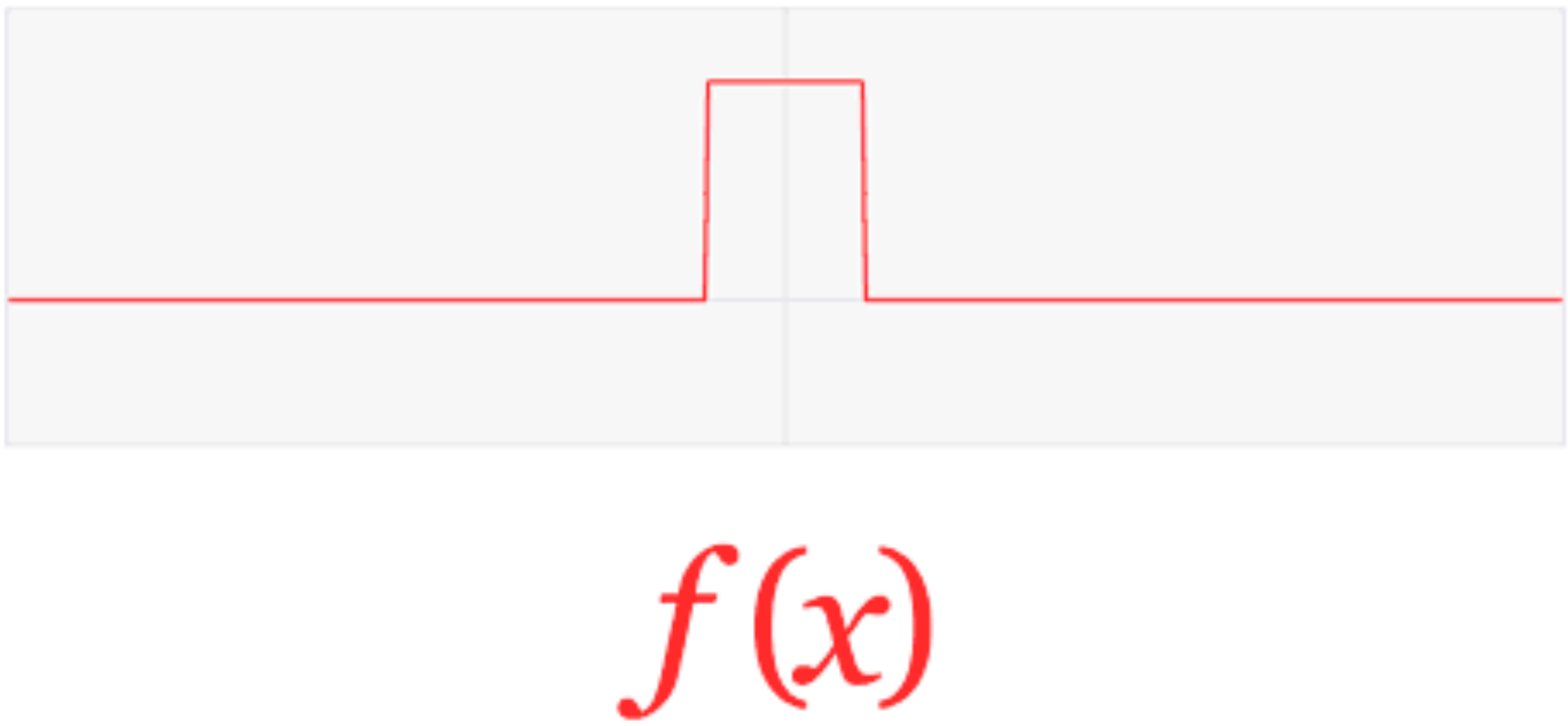
$$g(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi ift} g(t)\, dt e^{2\pi ift}\, df$$

FOURIER TRANSFORM AND ITS INVERSE

$$G(f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} g(t) dt$$

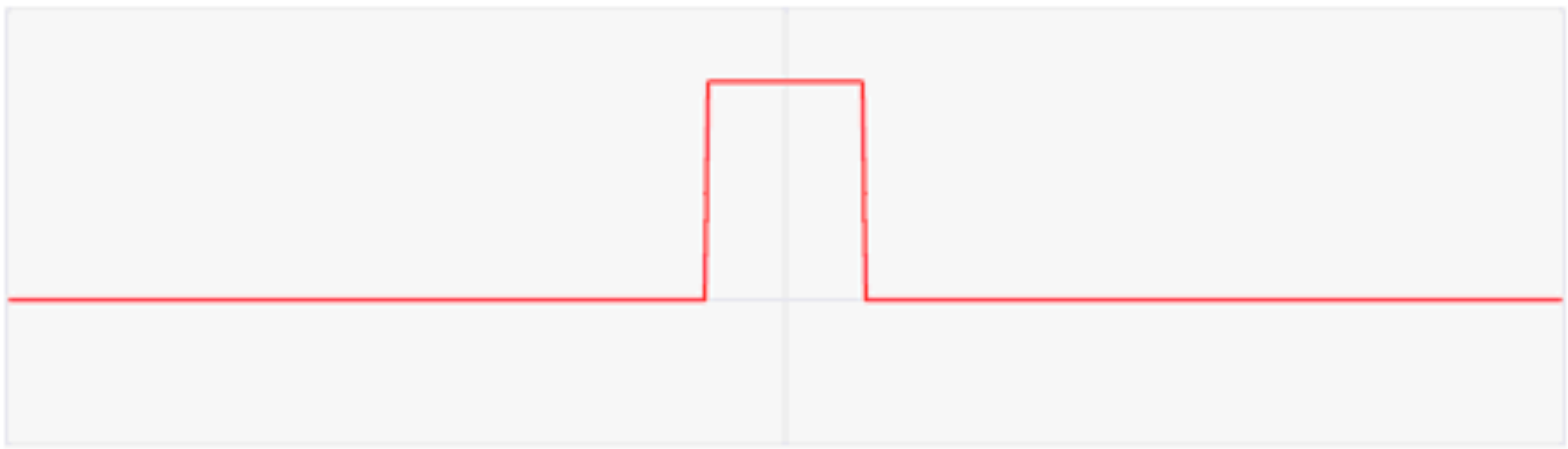
$$g(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} G(f) df$$

FOURIER TRANSFORM VISUALIZATION



https://en.wikipedia.org/wiki/File:Continuous_Fourier_transform_of_rect_and_sinc_functions.gif

FOURIER TRANSFORM VISUALIZATION



$f(x)$

https://en.wikipedia.org/wiki/File:Continuous_Fourier_transform_of_rect_and_sinc_functions.gif