MORE ON FOURIER SERIES

YU / HUGHES

SEPTEMBER 21-23, 2021

UMASS AMHERST

COMPLEX WAVES: REVIEW

BUILDING UP A SQUARE WAVE WITH SINES

https://en.wikipedia.org/wiki/User:LucasVB/Gallery#/media/File:Synthesis_square.gif

BUILDING UP A SQUARE WAVE WITH SINES

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BUILDING UP A SQUARE WAVE

https://en.wikipedia.org/wiki/User:LucasVB/Gallery#/media/File:Fourier_transform_time_and_frequency_domains.gif

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TWO REPRESENTATIONS OF SQUARE WAVE

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TWO REPRESENTATIONS OF SQUARE WAVE

https://en.wikipedia.org/wiki/User:LucasVB/Gallery#/media/File:Fourier_transform_time_and_frequency_domains.gif

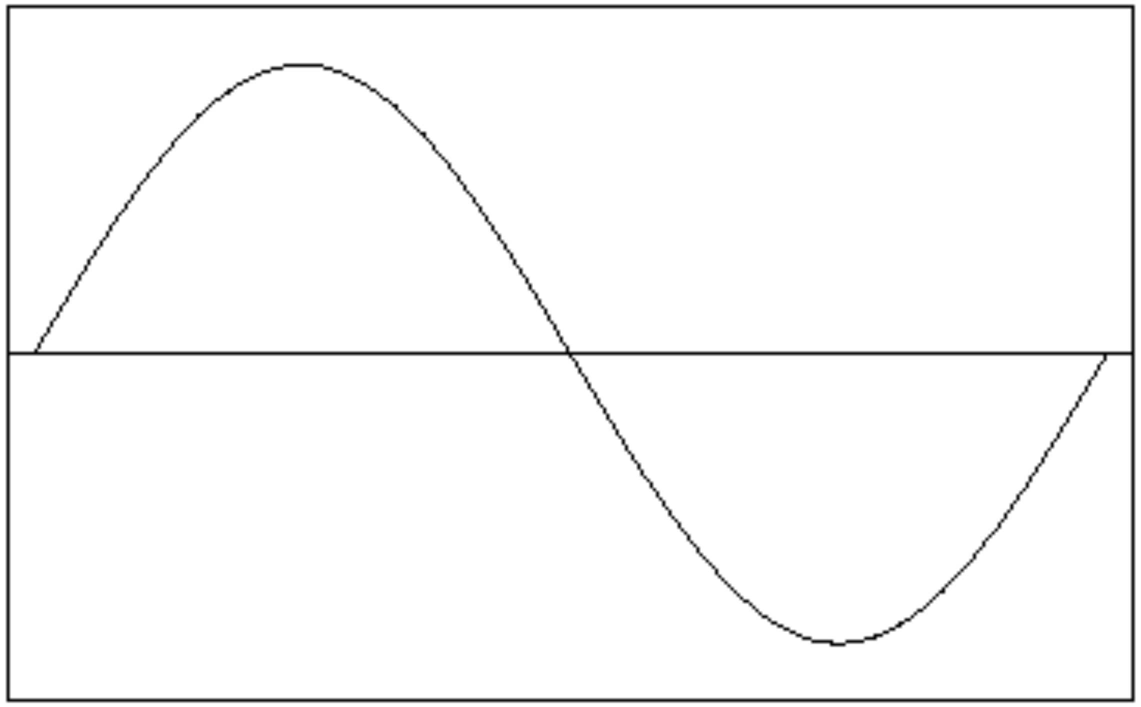
TWO REPRESENTATIONS OF SQUARE WAVE

Time domain: waveform

Frequency domain: spectrum

https://en.wikipedia.org/wiki/User:LucasVB/Gallery#/media/File:Fourier_transform_time_and_frequency_domains.gif

HARMONIC AMPLITUDES



https://sites.google.com/a/georgiasouthern.edu/julia-inozemtseva/teaching-math-animations-and-pics?tmpl=%2Fsystem%2Fapp%2Ftemplates%2Fprint%2F&showPrintDialog=1#TOC-Fourier-Series

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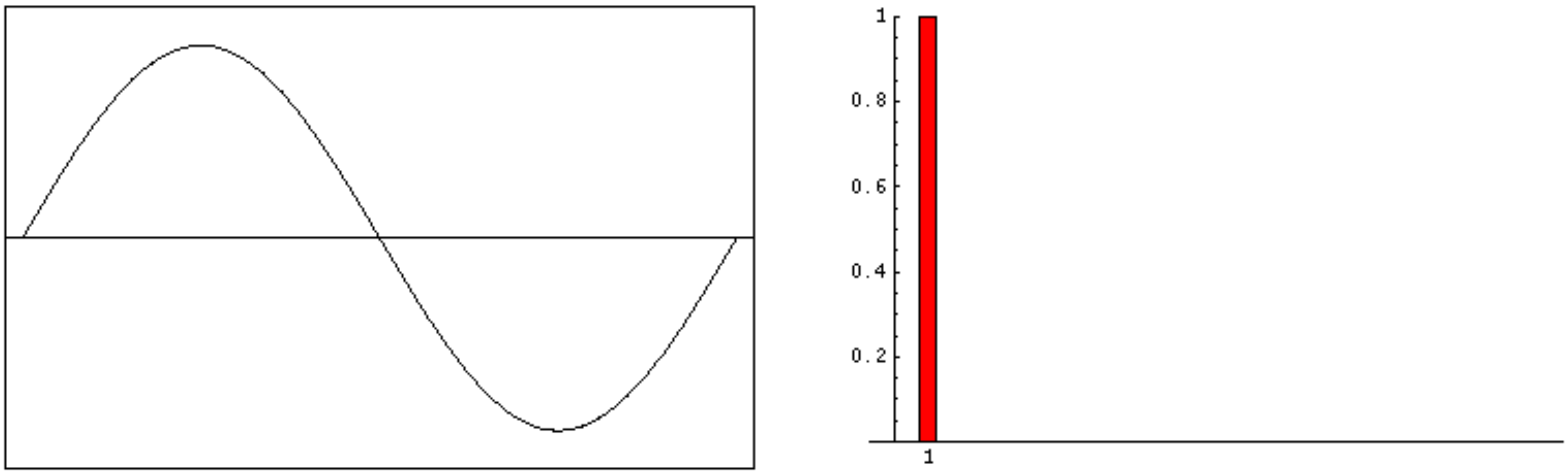

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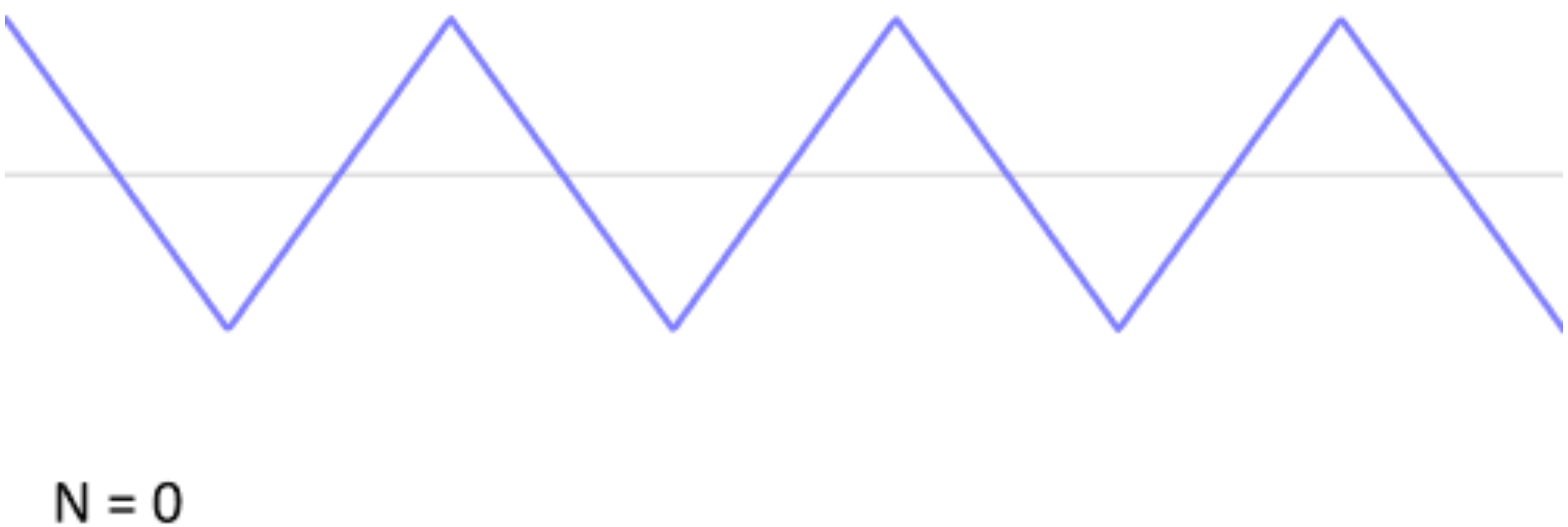
BUILDING UP A TRIANGLE WAVE WITH SINES

N = 0

https://en.wikipedia.org/wiki/User:LucasVB/Gallery#/media/File:Fourier_transform_time_and_frequency_domains.gif

BUILDING UP A TRIANGLE WAVE WITH SINES



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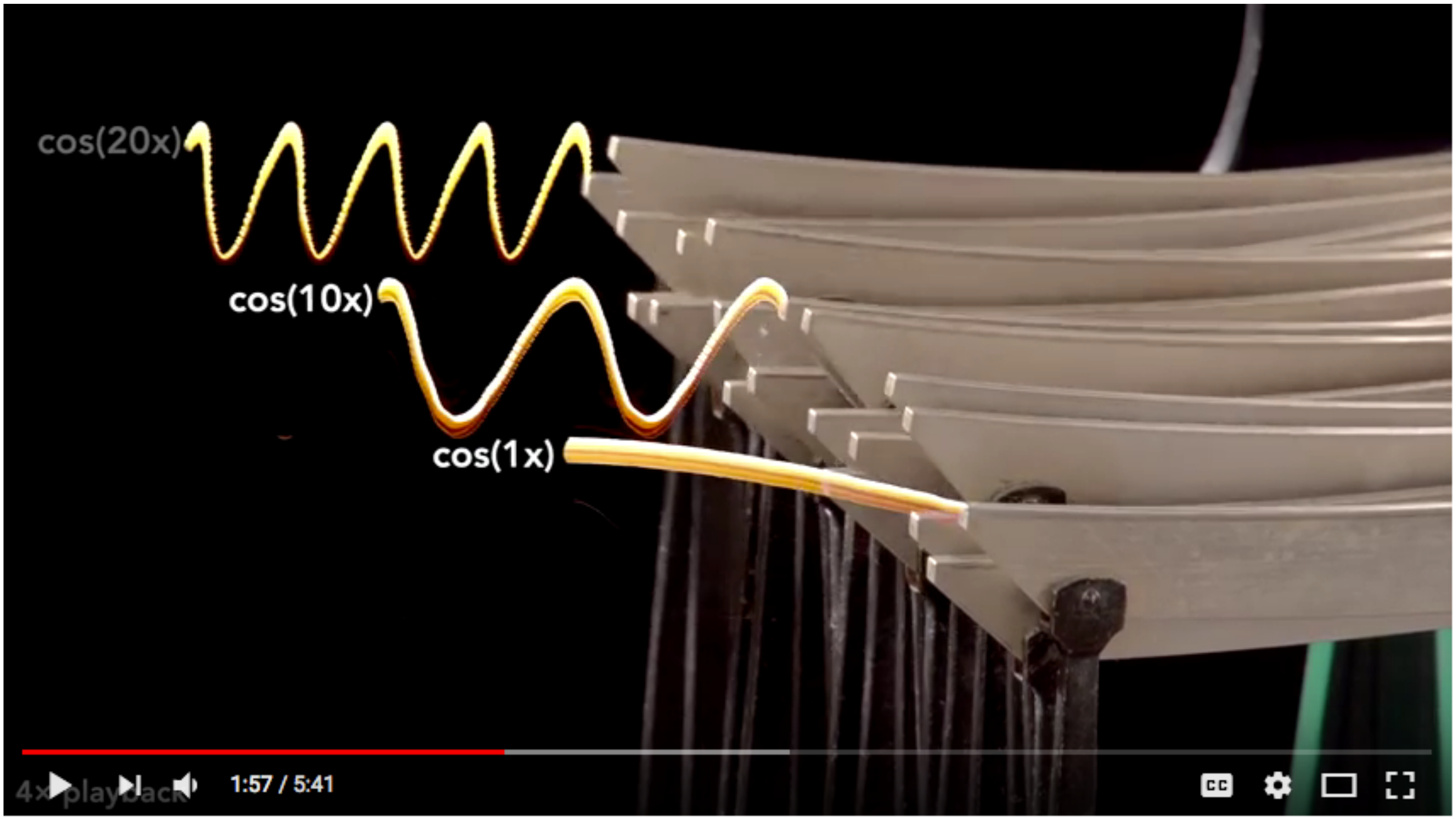
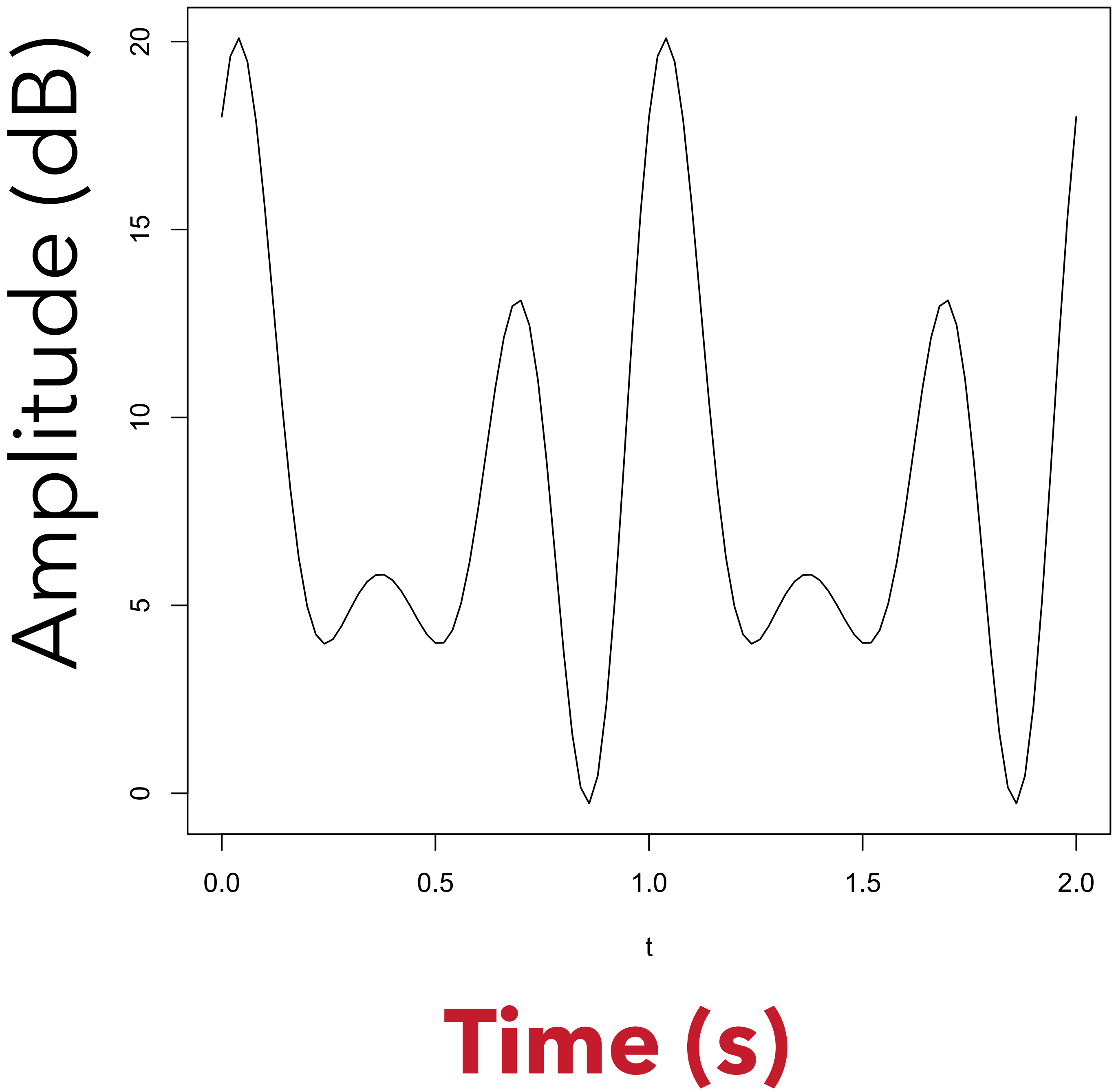
COMPLEX WAVES IN THE 19TH CENTURY

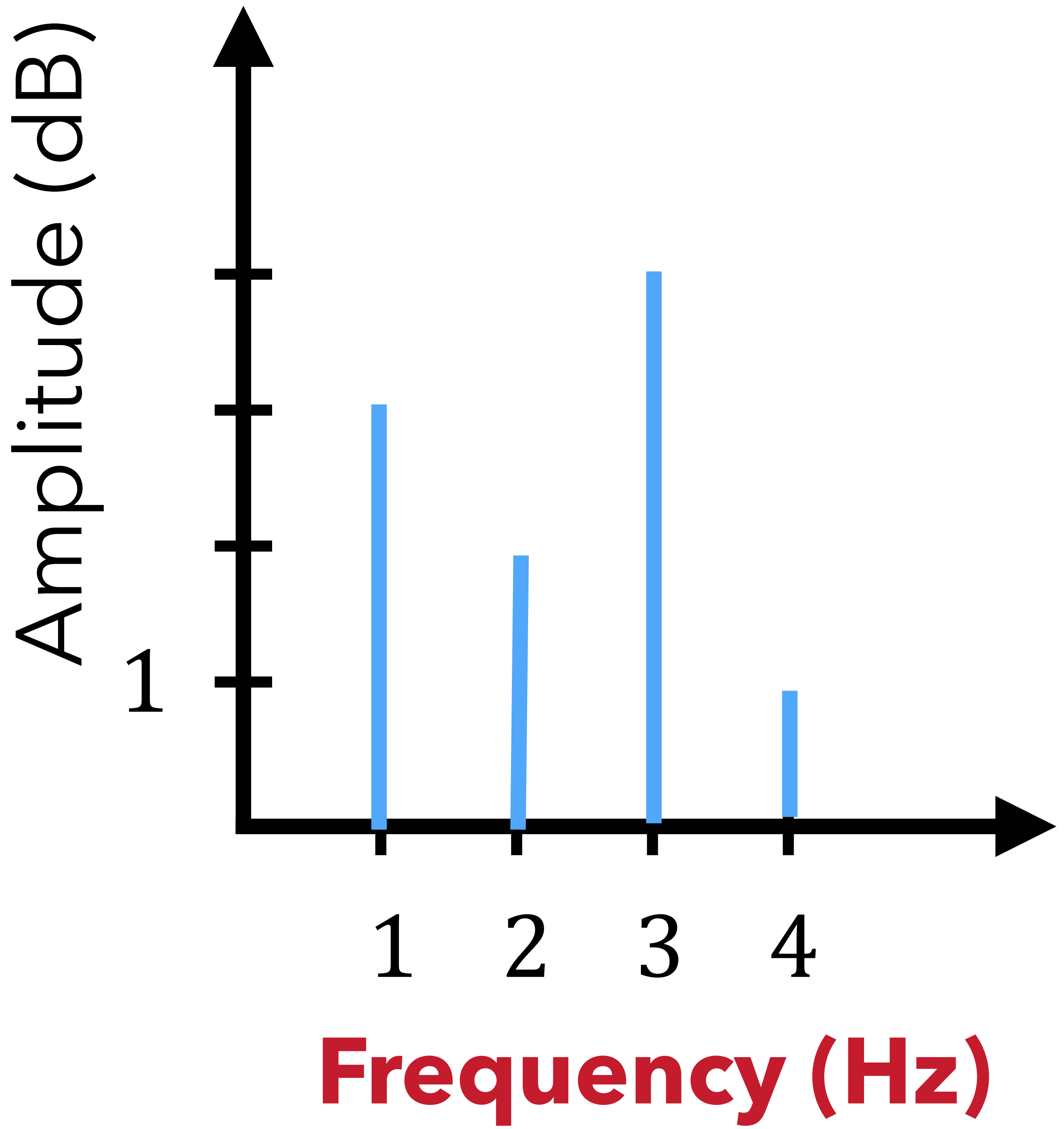


https://www.youtube.com/watch?v=NAsM30MAHLg

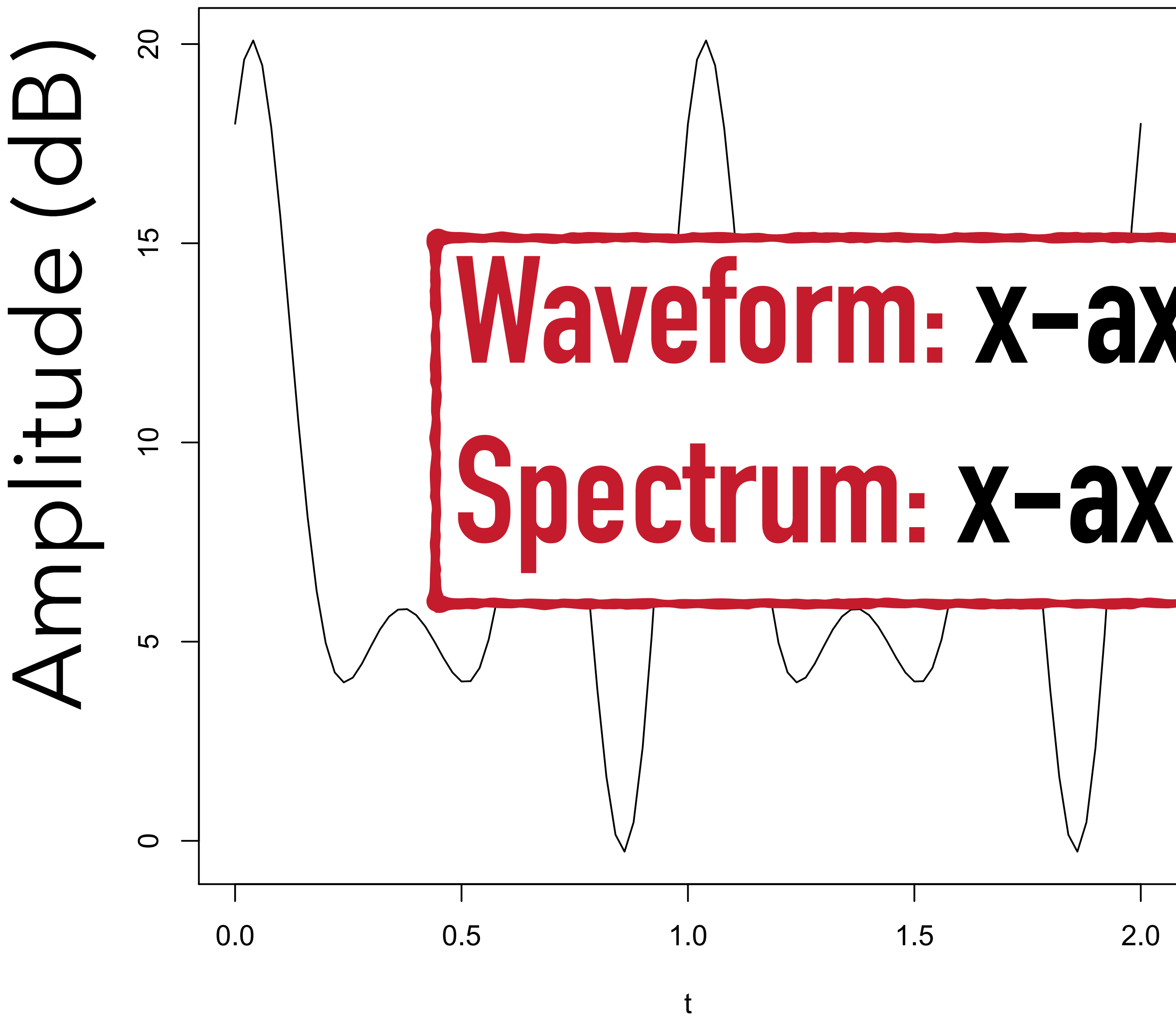
THE SPECTRUM AND THE SPECTROGRAM

SPECTRUM OF COMPLEX WAVEFORM

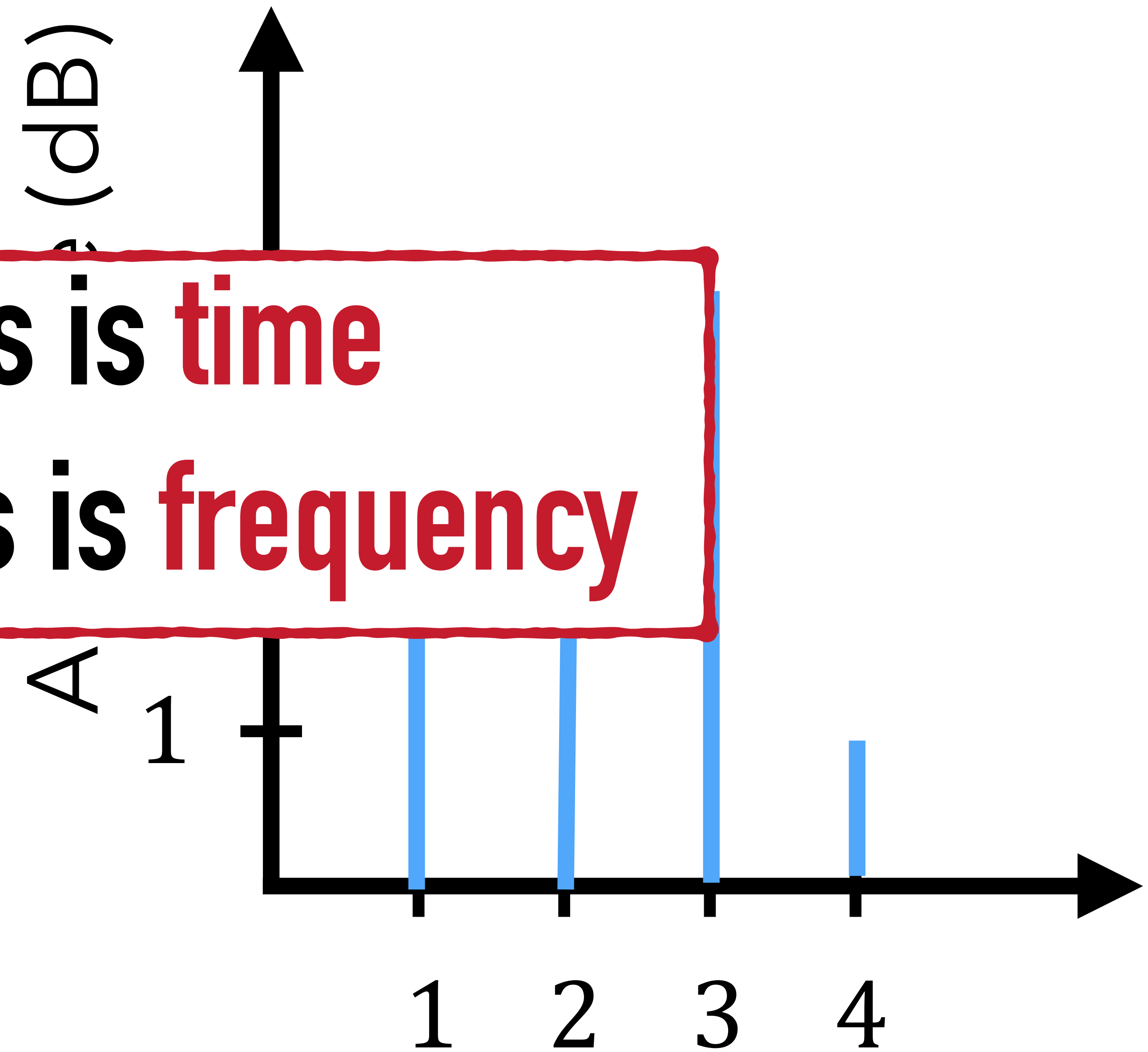




SPECTRUM OF COMPLEX WAVEFORM



Time (s)

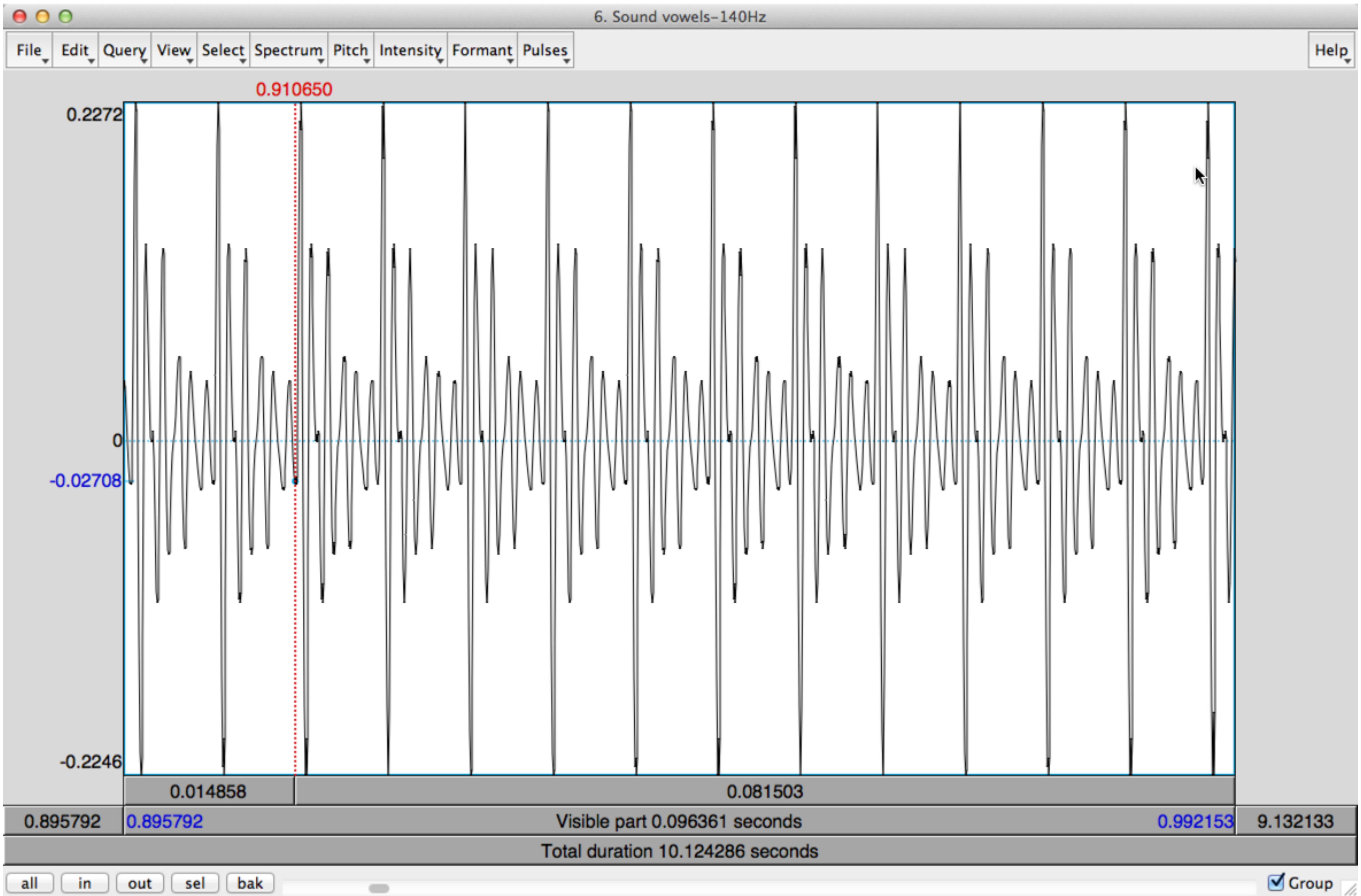


Frequency (Hz)

Waveform: x-axis is time

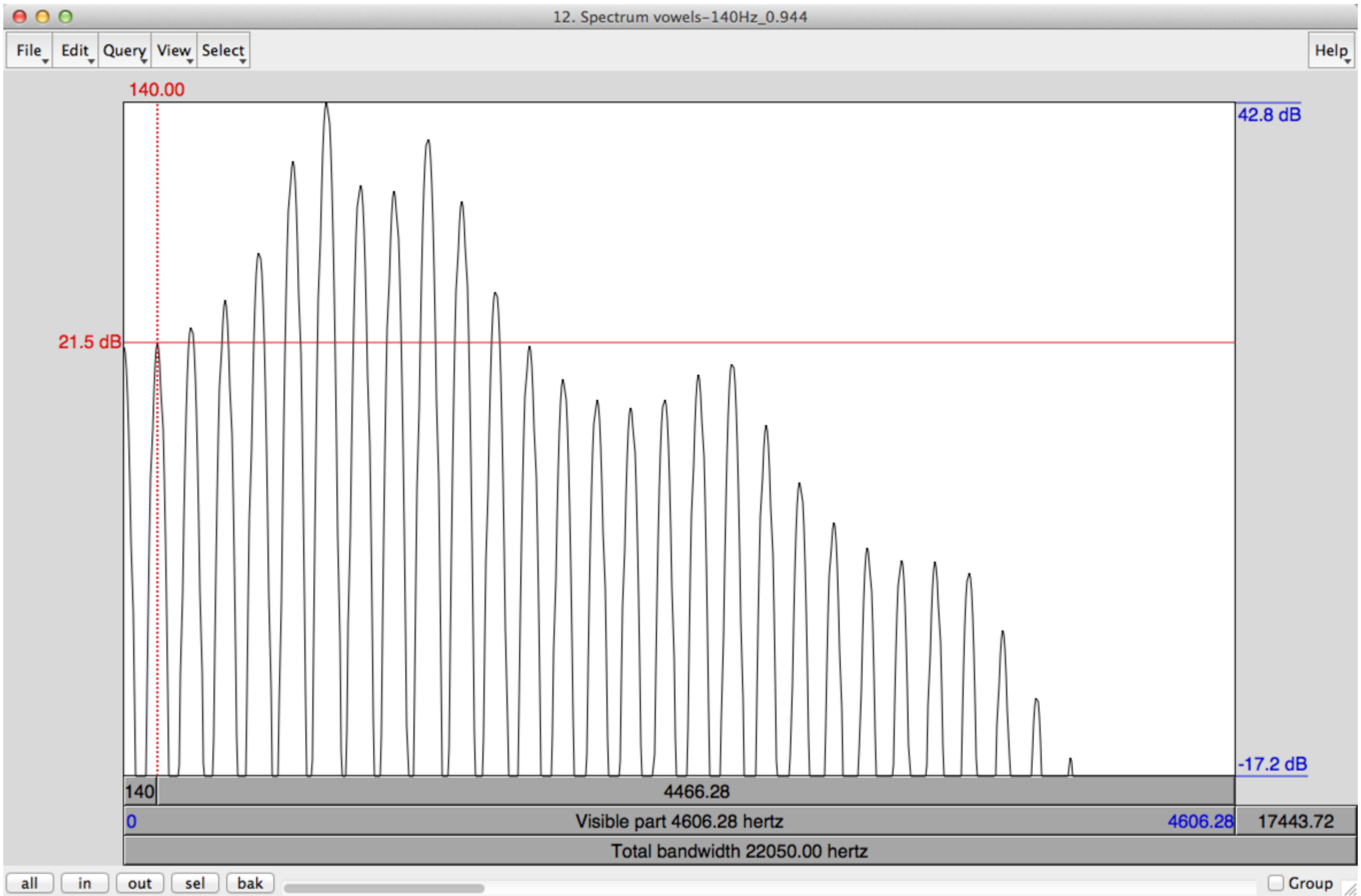
Spectrum: x-axis is frequency

IN PRAAT: FROM WAVEFORM...

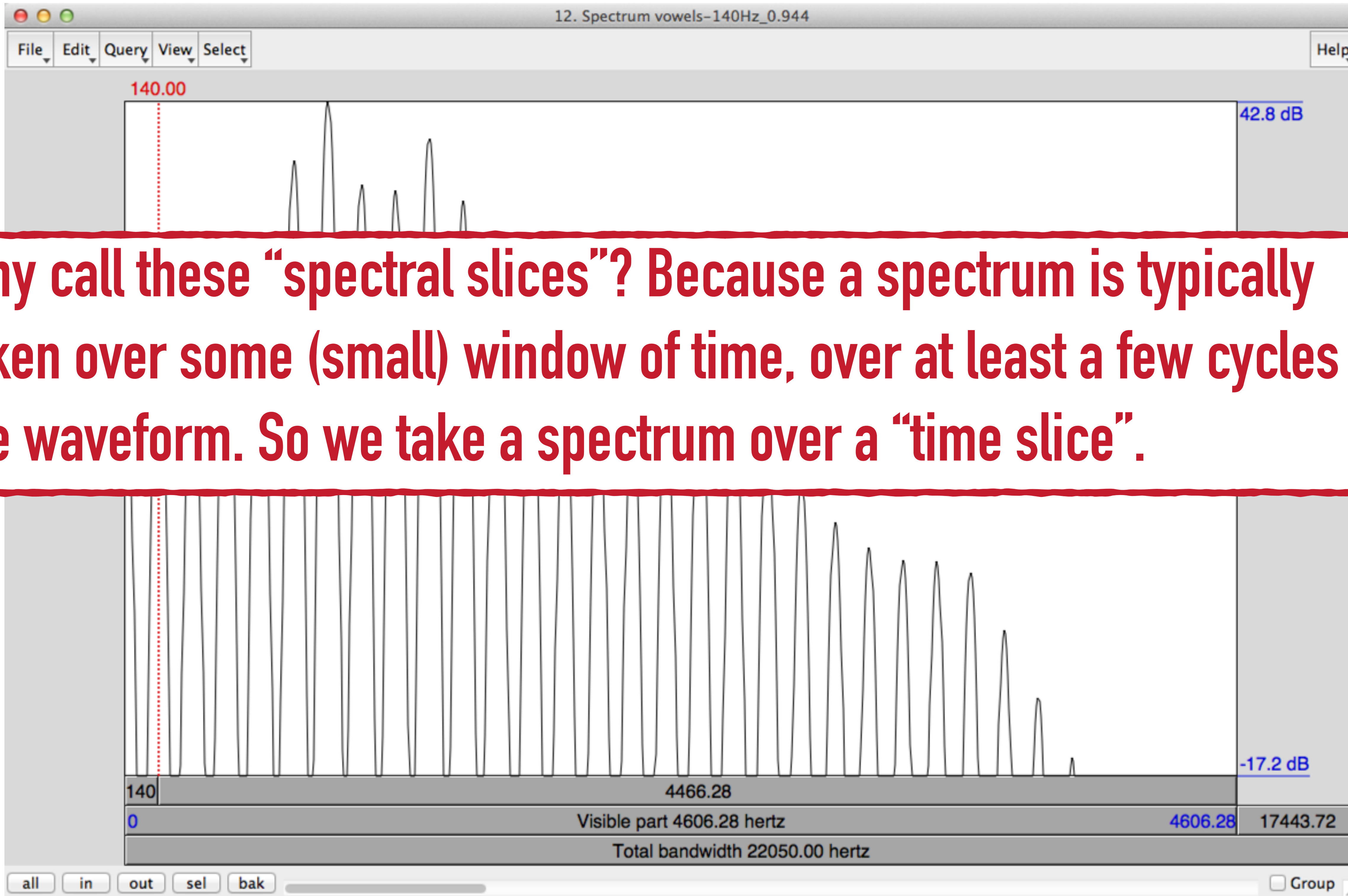


...TO SPECTRUM (SPECTRAL SLICE)

…TO SPECTRUM (SPECTRAL SLICE)

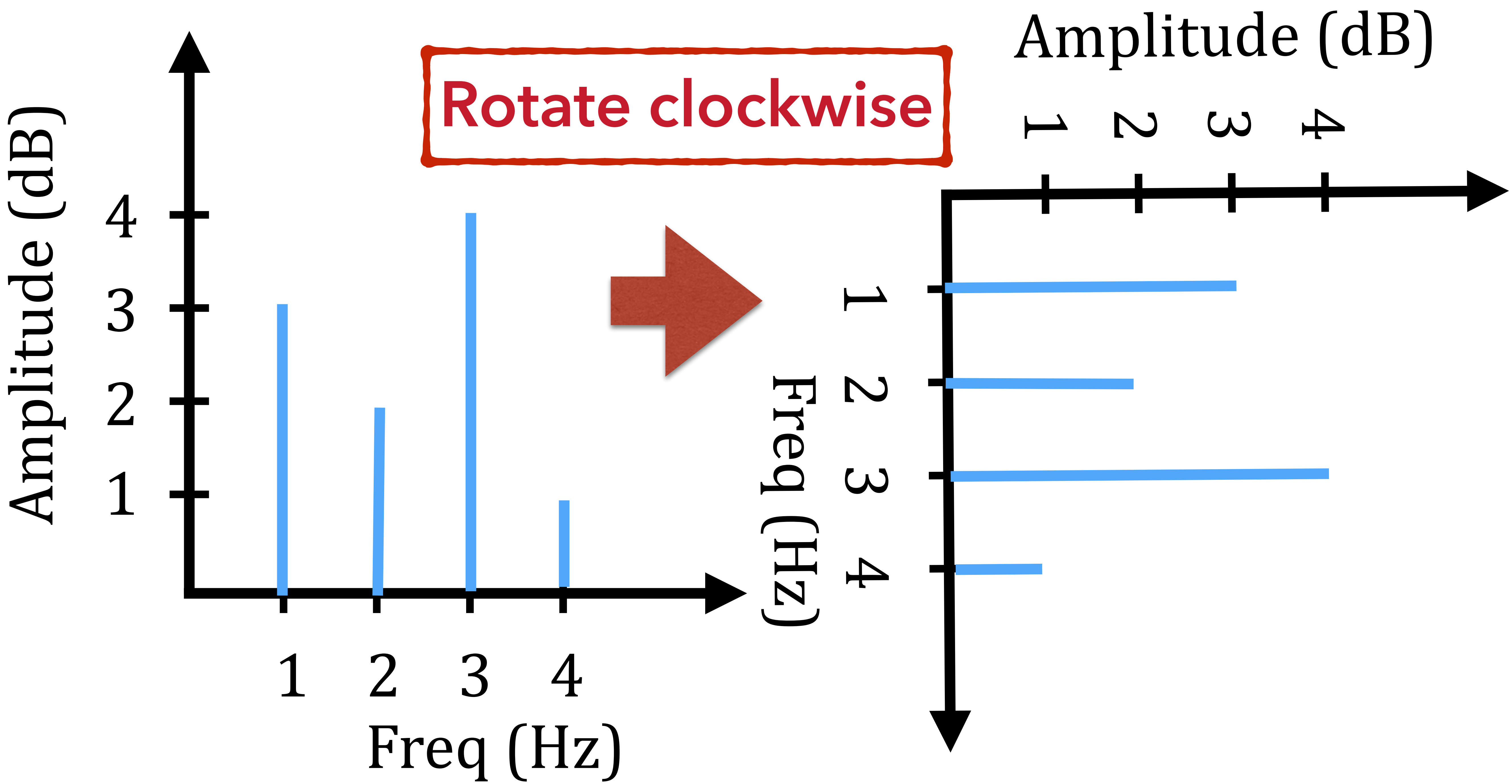


...TO SPECTRUM (SPECTRAL SLICE)

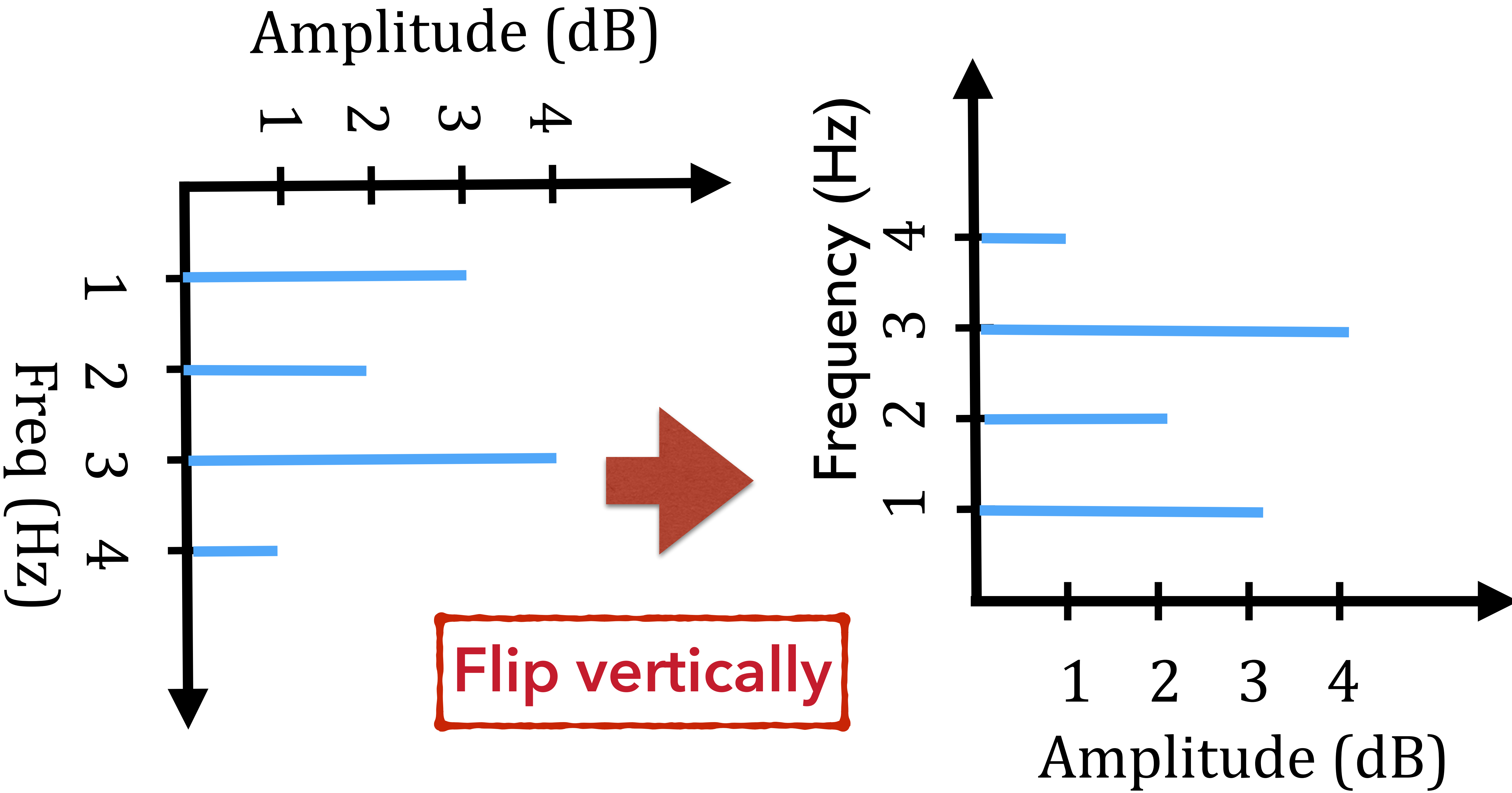


Why call these "spectral slices"? Because a spectrum is typically taken over some (small) window of time, over at least a few cycles of the waveform. So we take a spectrum over a "time slice".

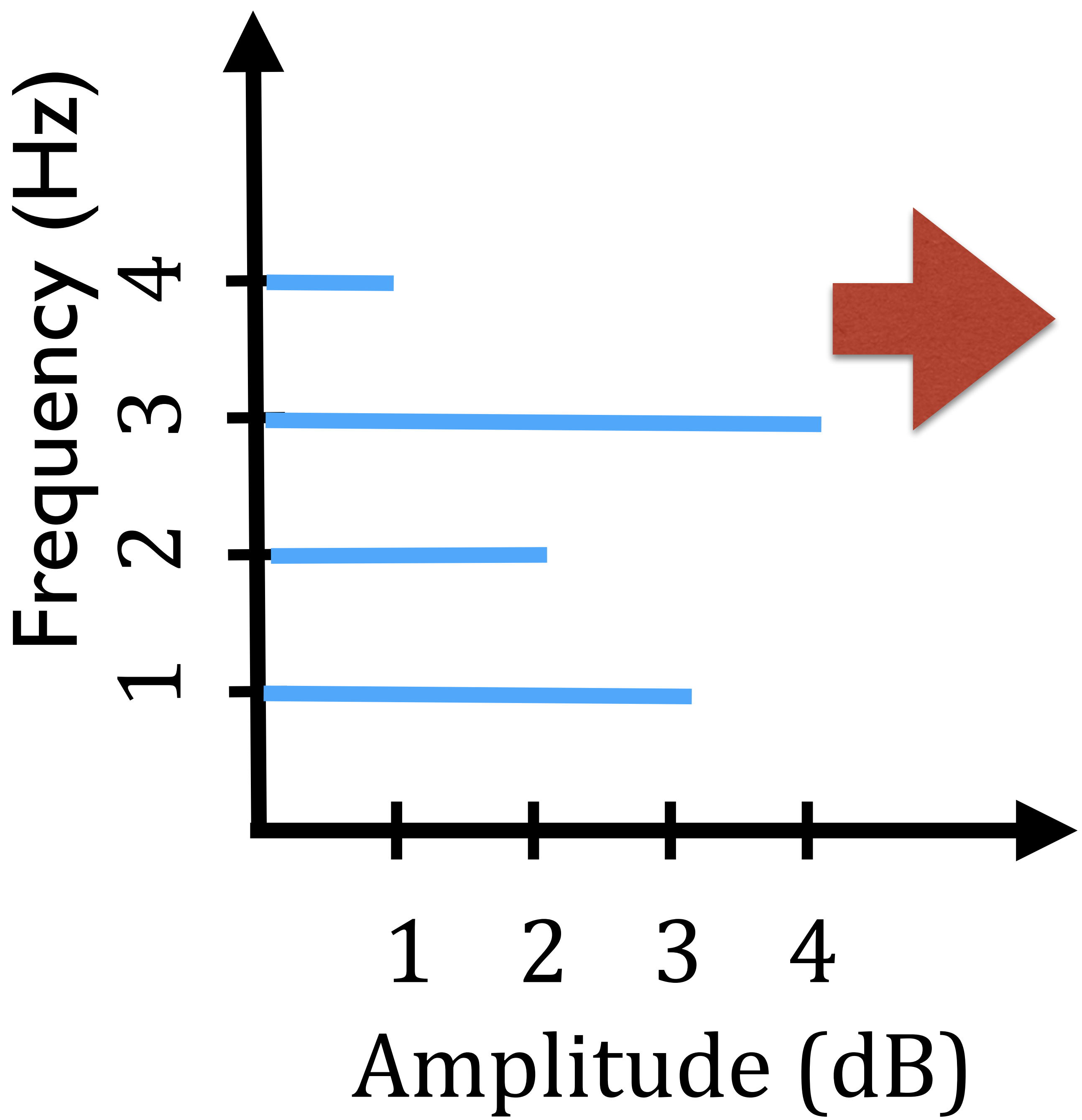
FROM SPECTRUM TO SPECTROGRAM: STEP 1



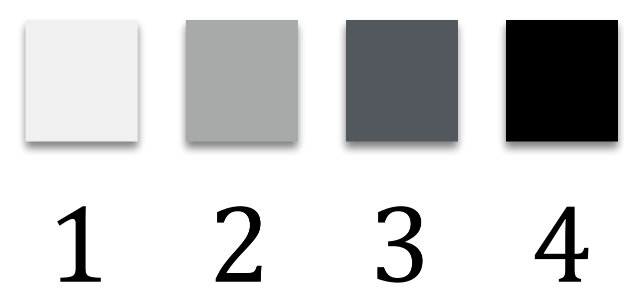
FROM SPECTRUM TO SPECTROGRAM: STEP 2



FROM SPECTRUM TO SPECTROGRAM: STEP 3

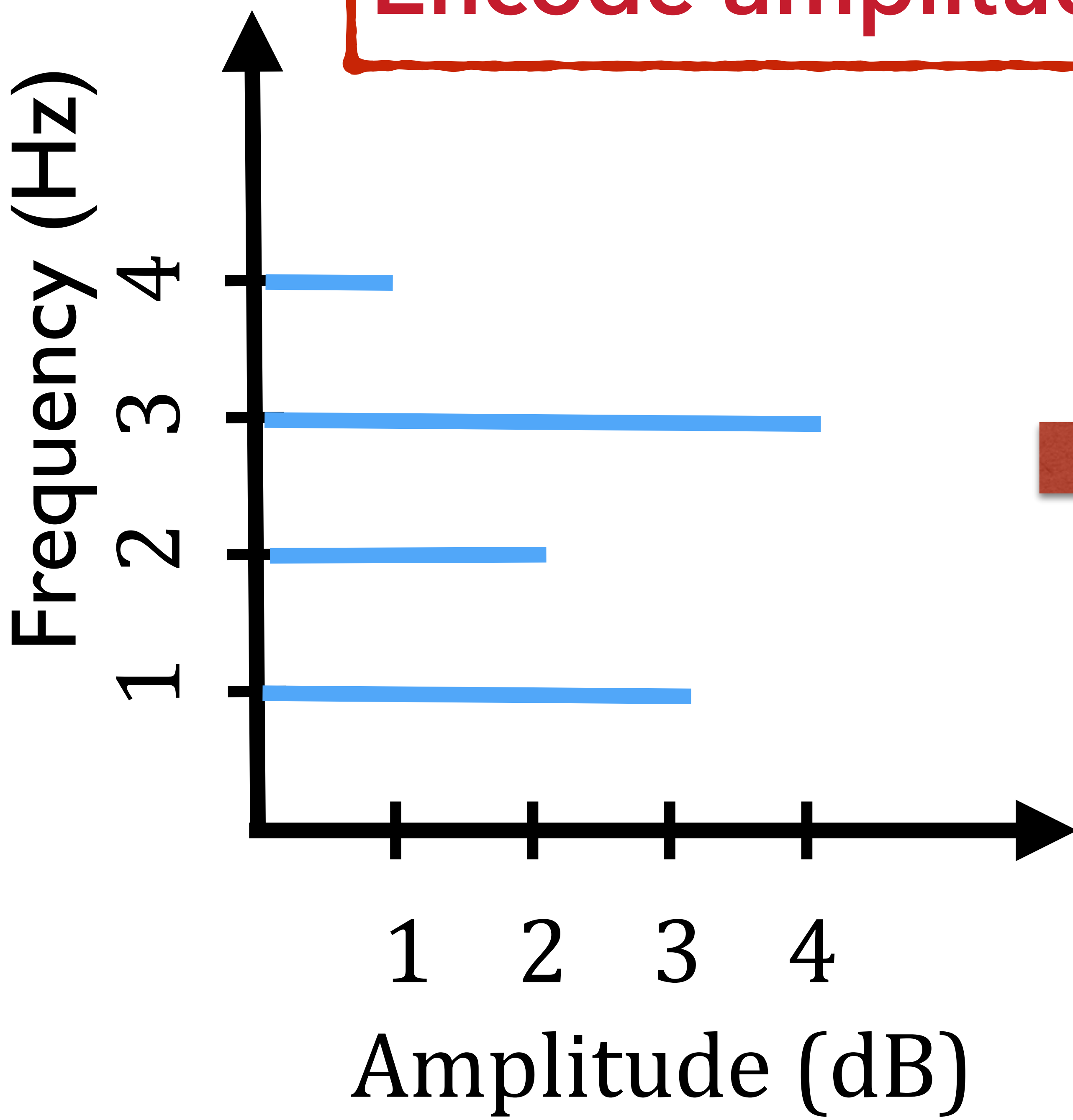


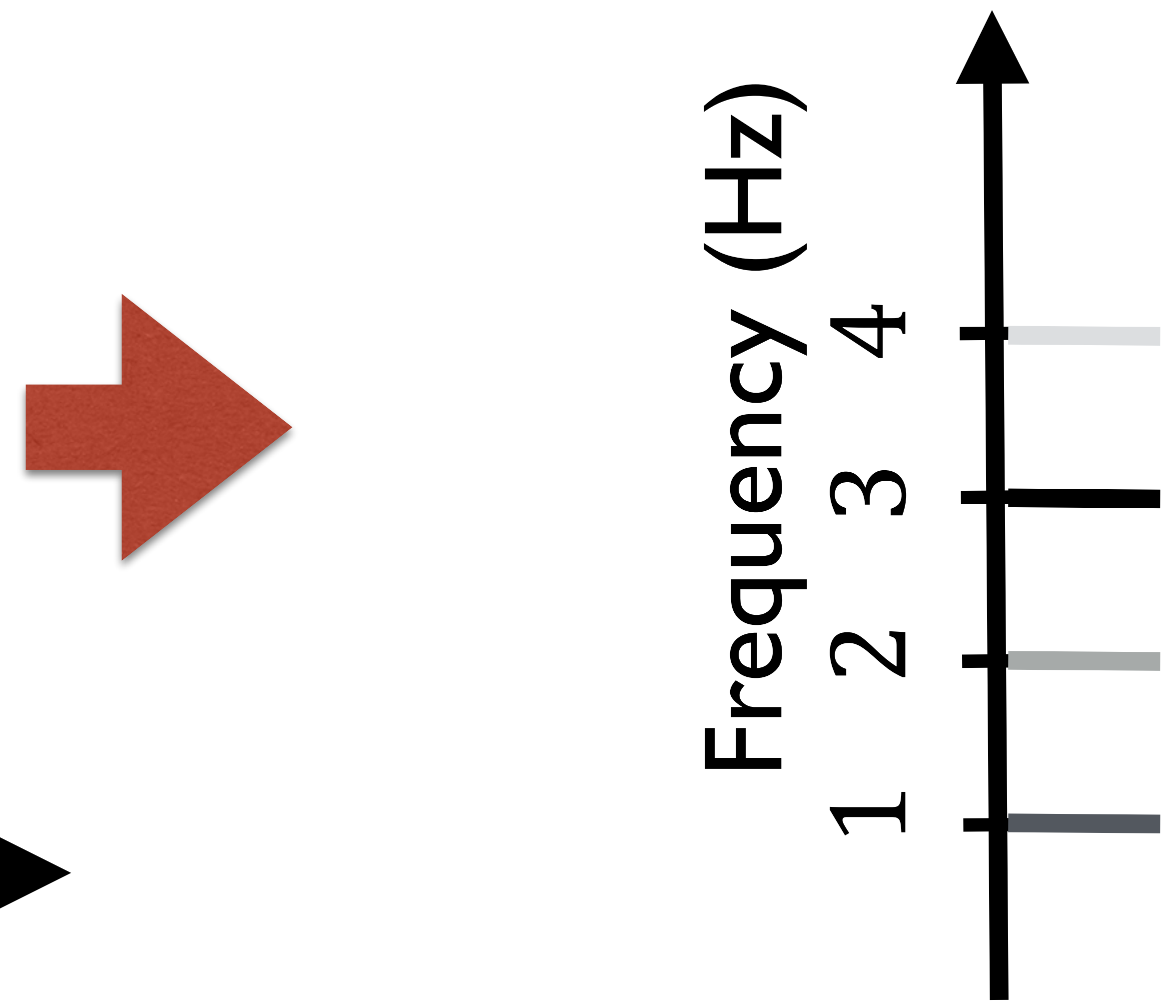
Encode amplitude levels (the height/ length of the blue spikes) in grayscale



FROM SPECTRUM TO SPECTROGRAM: STEP 4

Encode amplitudes in grayscale…

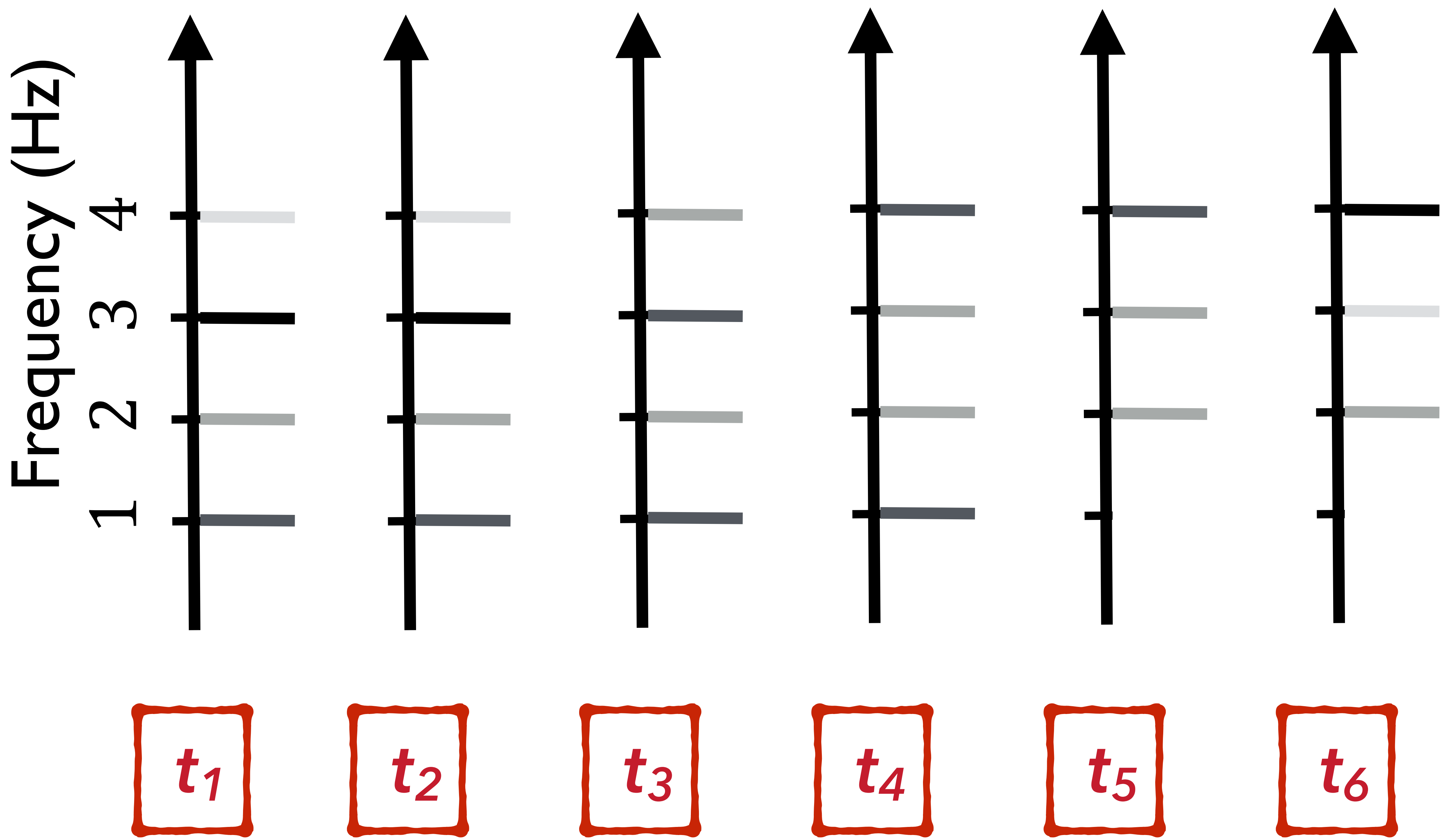




…at time t_1

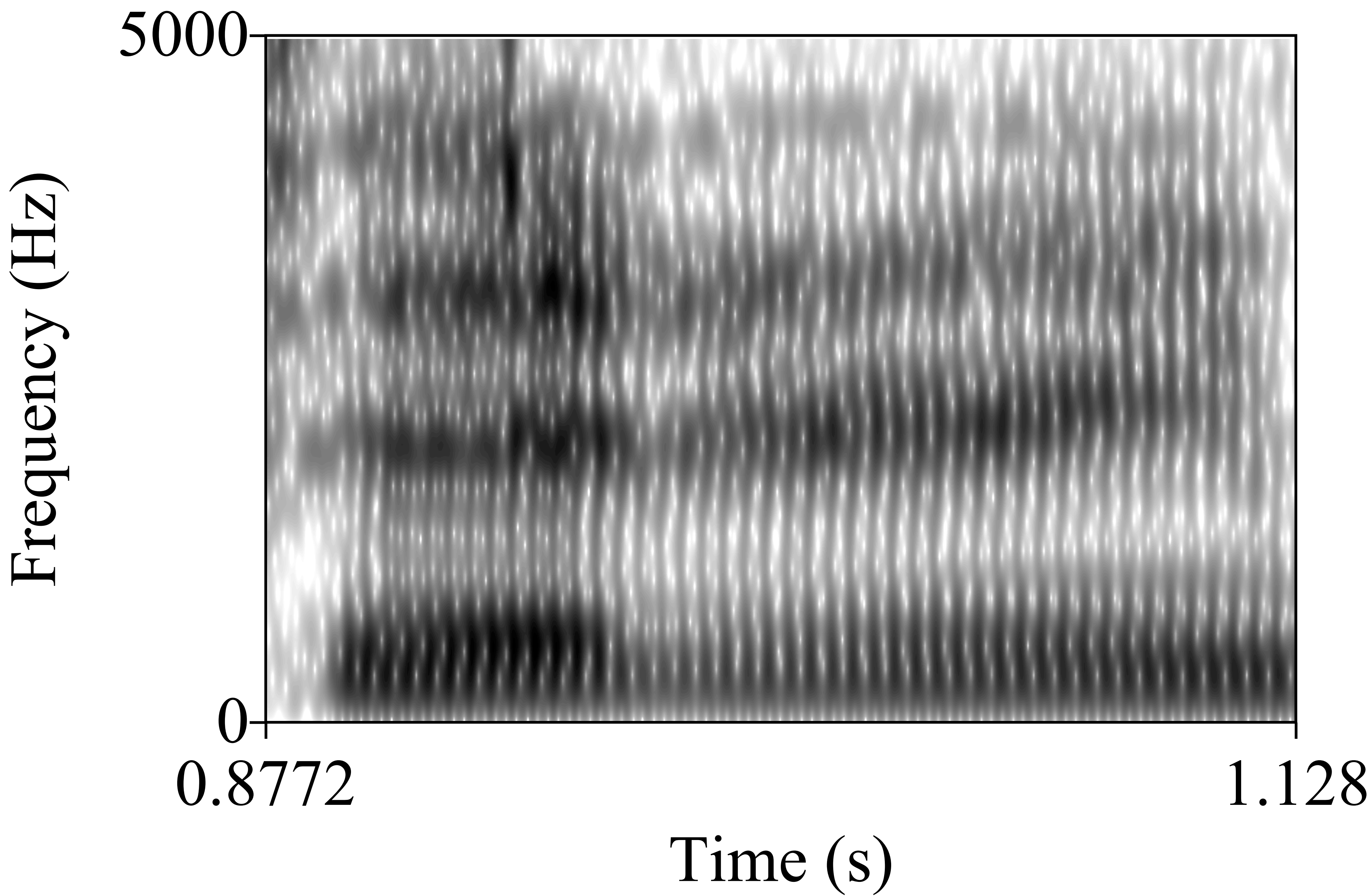
FROM SPECTRUM TO SPECTROGRAM: STEP 5

Line up spectral slices taken over time…



FROM SPECTRUM TO SPECTROGRAM: FINIS

...and you have a spectrogram



EXERCISE: SPECTRAL SLICES IN PRAAT

1. Open the files `tone-140Hz.wav` and `vowels-140Hz.wav` here:
   https://drive.google.com/open?id=1Le4hfR1KQqQEQXGbXDdV0iMeALZIG4ow

2. Zoom into selection in the waveforms of each of the files, with at least a handful of cycles visible in the selection window. You will be taking spectral slices over the portion of the waveform in the selection windows.

3. For each waveform, take a spectral slice
   - To take a spectral slice, go to Spectrum > View Spectral Slice, or Cmd (Ctrl)-L.
   - For the vowels file, you can take a spectral slice from each of the three vowels

4. Compare the spectral slices of the two different files
   - What frequencies are present? Why?

EXERCISE: ADDITIVE SYNTHESIS 1

- **In a jupyter notebook, define the following component sinusoidal waves (use a sampling rate of 22050 Hz, and make the duration somewhere between 1 and 3 seconds):**
  - $cos(2\pi t)$, $sin(2\pi t)$
  - $cos(4\pi t)$, $sin(4\pi t)$
  - $cos(6\pi t)$, $sin(6\pi t)$
- ***What is the period of each of the component waves? What is the period of the complex wave created by adding together all the component waves? Why? Calculate the periods by hand, and also look visually at plots in jupyter notebook, at what happens as you add in additional terms in the Fourier series.***

EXERCISE II: ADDITIVE SYNTHESIS

- **Now do the same exercise, but for the following component waves (could this be a Fourier series?):**
    - $cos(\pi t)$
    - $sin(2\pi t)$
    - $cos(5\pi t)$
    - $sin(8\pi t)$
- ***What is the period of each of the component waves? What is the period of the complex wave created by adding together all the component waves? Why? Calculate the periods by hand, and also look visually at plots in jupyter notebook, at what happens as you add in additional terms in the Fourier series.***

FOURIER COEFFICIENTS

FOURIER SERIES

$$a_0 + \sum_{n=1}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

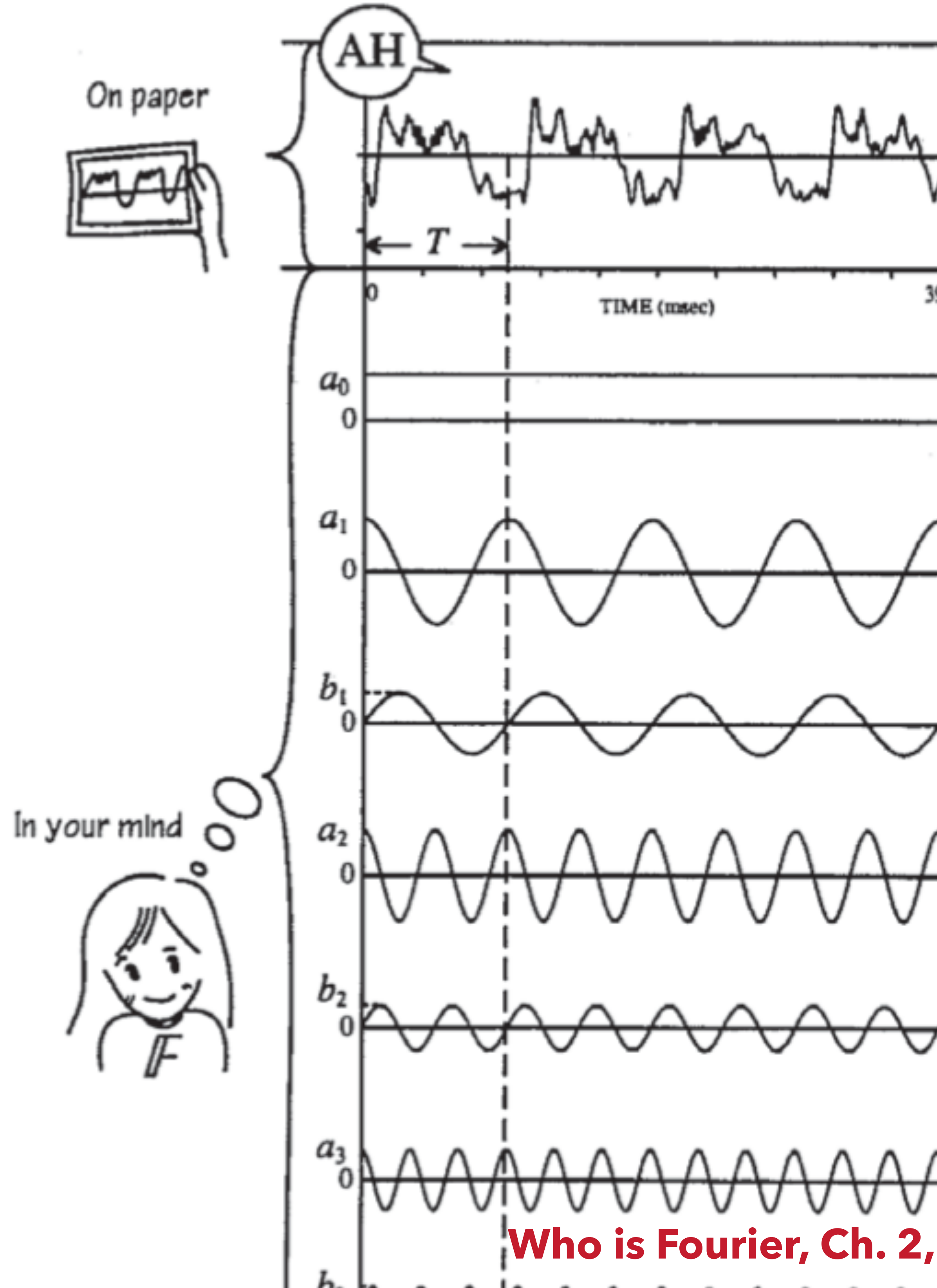
FOURIER COEFFICIENTS

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 2\pi ntdt$$

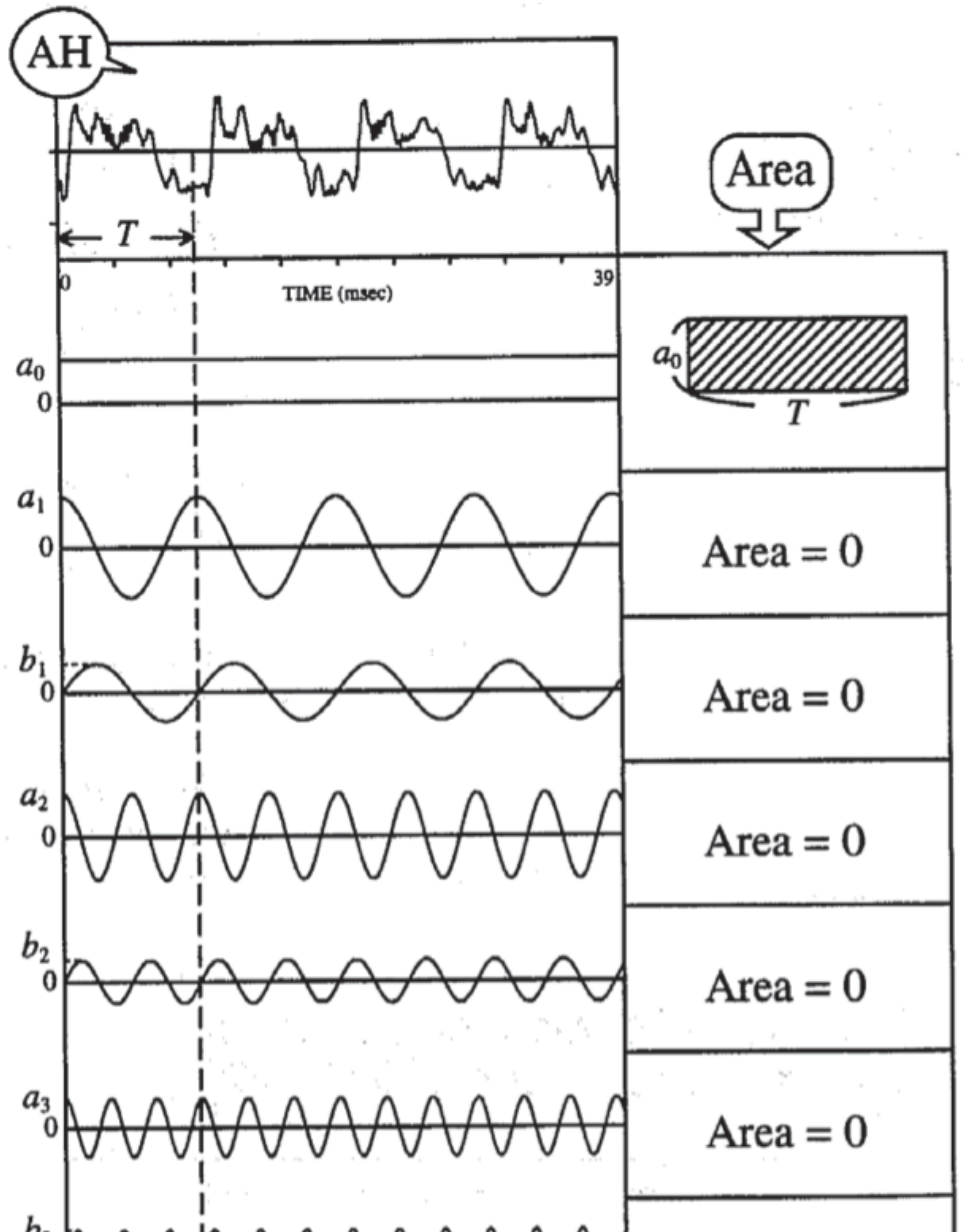
CACULATING THE 0TH TERM

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$



Who is Fourier, Ch. 2, p. 82

CACULATING THE 0TH TERM



Who is Fourier, Ch. 2, p. 83

FOURIER SERIES INTEGRAL IDENTITIES

https://commons.wikimedia.org/wiki/File:Fourier_series_integral_identities.gif

FOURIER SERIES INTEGRAL IDENTITIES

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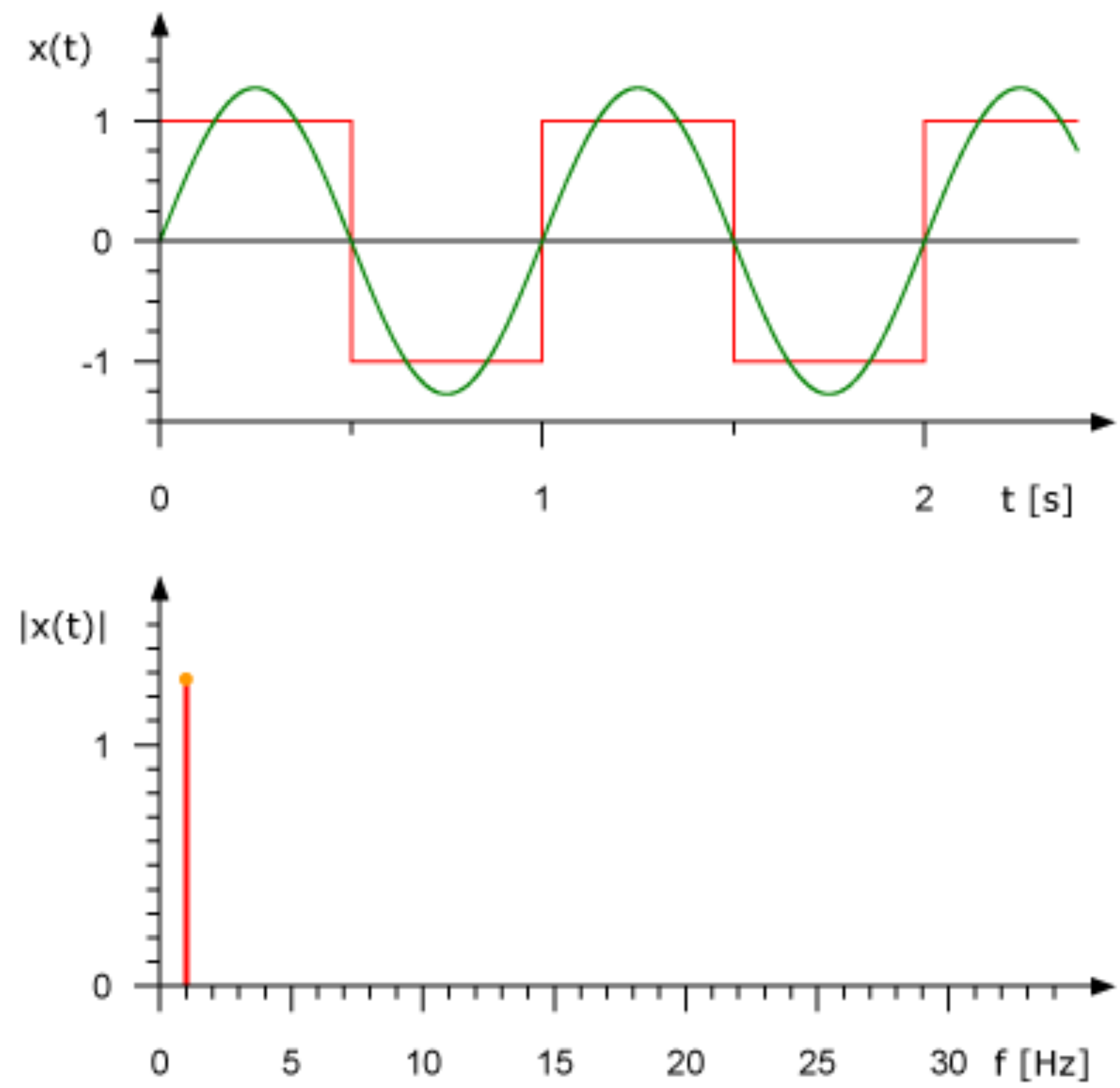
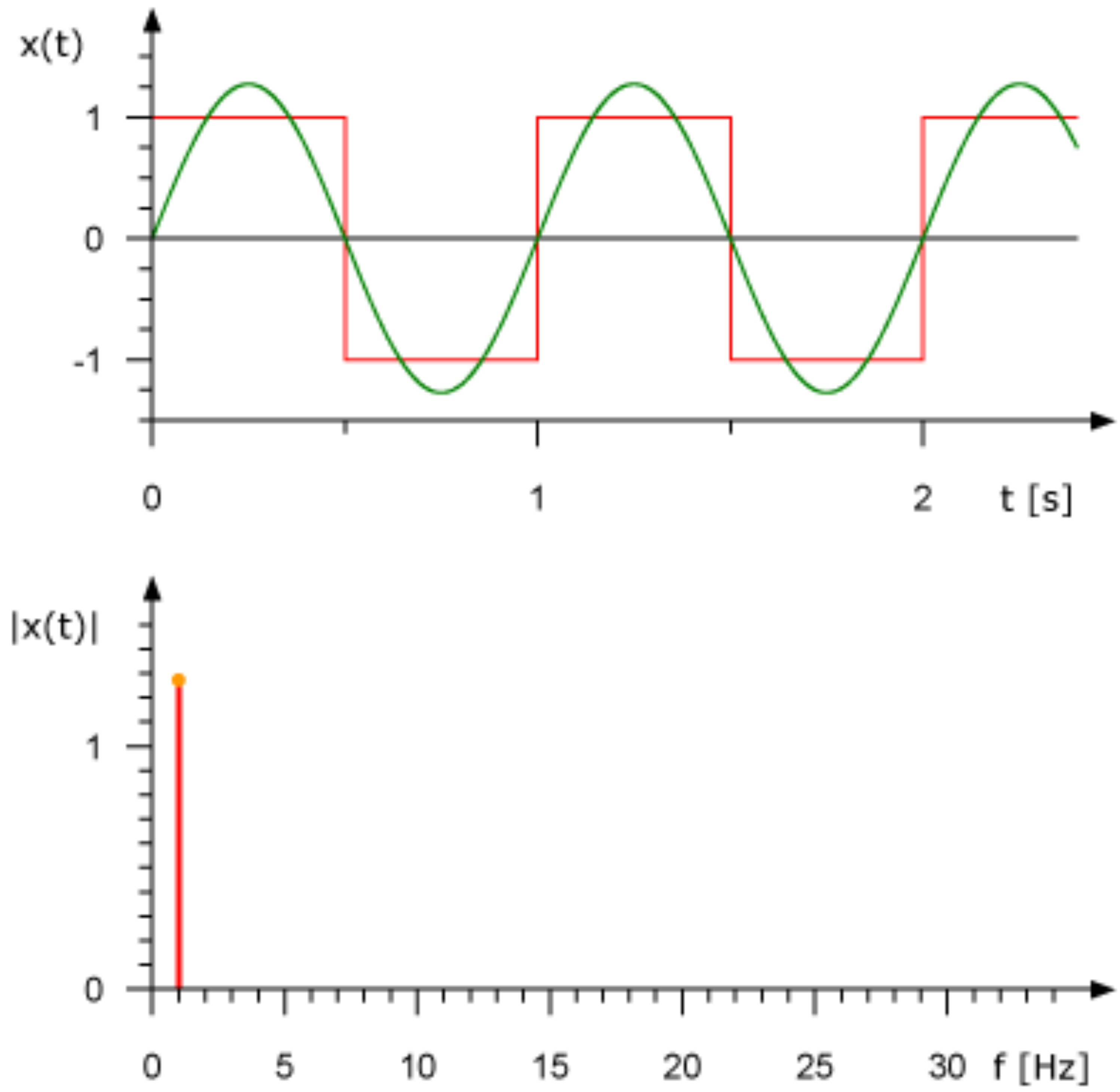
BACK TO SQUARE WAVES…



https://upload.wikimedia.org/wikipedia/commons/a/af/Fourier_synthesis_square_wave_animated.gif

BACK TO SQUARE WAVES...



https://upload.wikimedia.org/wikipedia/commons/a/af/Fourier_synthesis_square_wave_animated.gif

CALCULATE 0TH TERM FOR SQUARE WAVE

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

INCLUDING THE 0TH TERM IN THE SUM

0TH TERM FOLDED INTO SUM

$$\sum_{n=0}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 2\pi ntdt$$

0TH TERM FOLDED INTO SUM

$$\sum_{n=0}^{N} (a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

How is the a_0 term here related to the a_0 when we separated out a_0 as a term?

$$a_n = \frac{2}{T}\int_0^T f(t)\cos 2\pi ntdt$$

EXERCISE: ORTHOGONALITY

- Take the set of sinusoids that you defined and plotted in the first additive synthesis exercise
- Show that the Fourier series terms form an orthogonal basis set by plotting the product of the Fourier series terms, and also integrating the product of the terms.

COMPLEX REPRESENTATION OF FOURIER SERIES

WHERE WE'RE HEADED…

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi i n t}$$

$$C_n = \frac{1}{T} \int_0^T e^{-2\pi i n t} f(t)\, dt$$

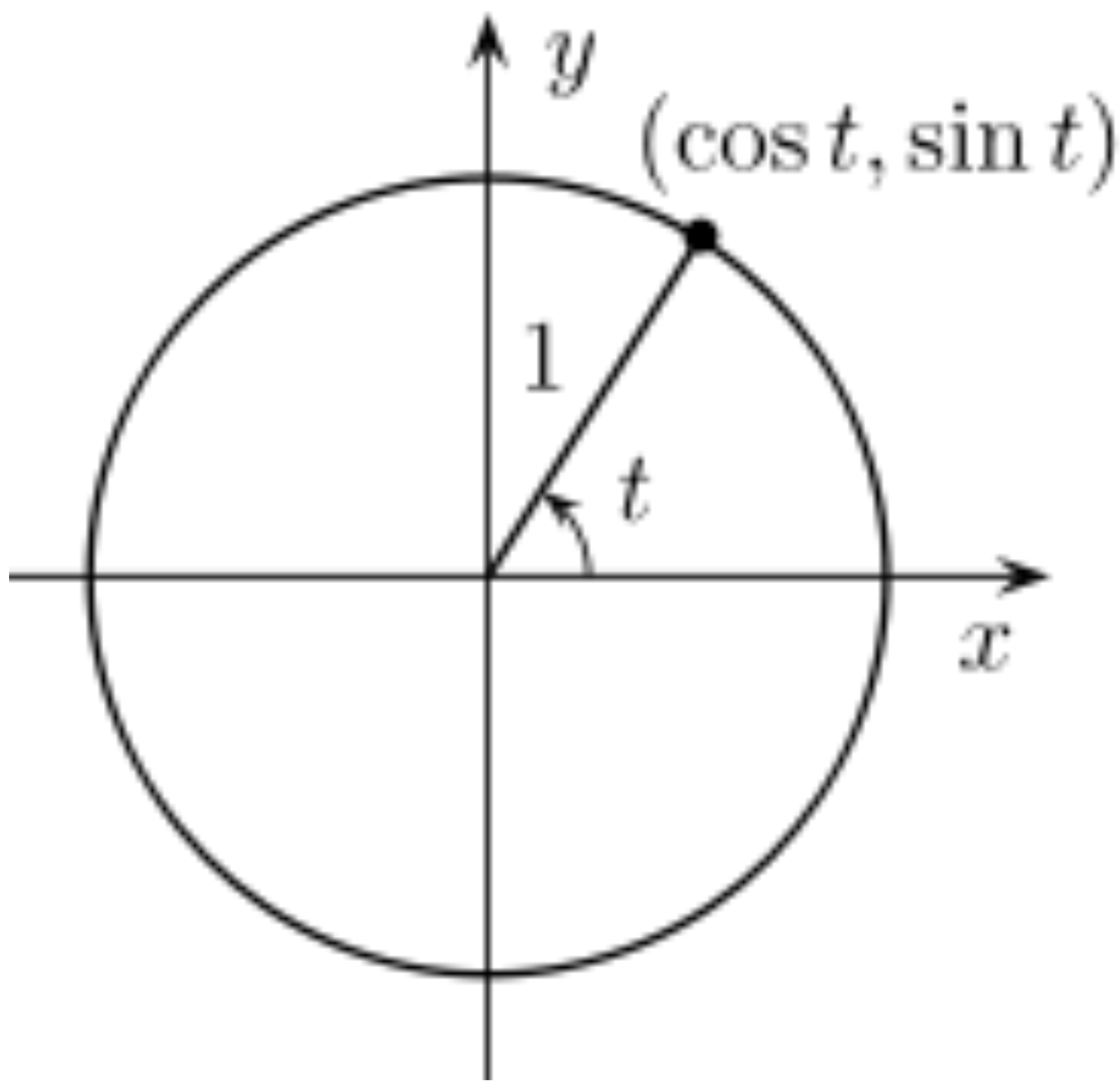
WHERE WE'RE HEADED…

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi i n t}$$

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi i n t} f(t)\,dt$$

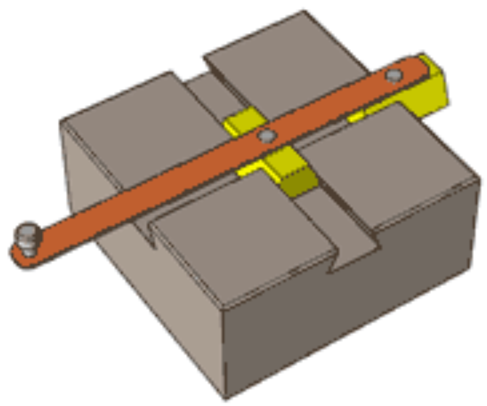
See Who is Fourier Chs. 10,11
Osgood notes Sec 1.6.1

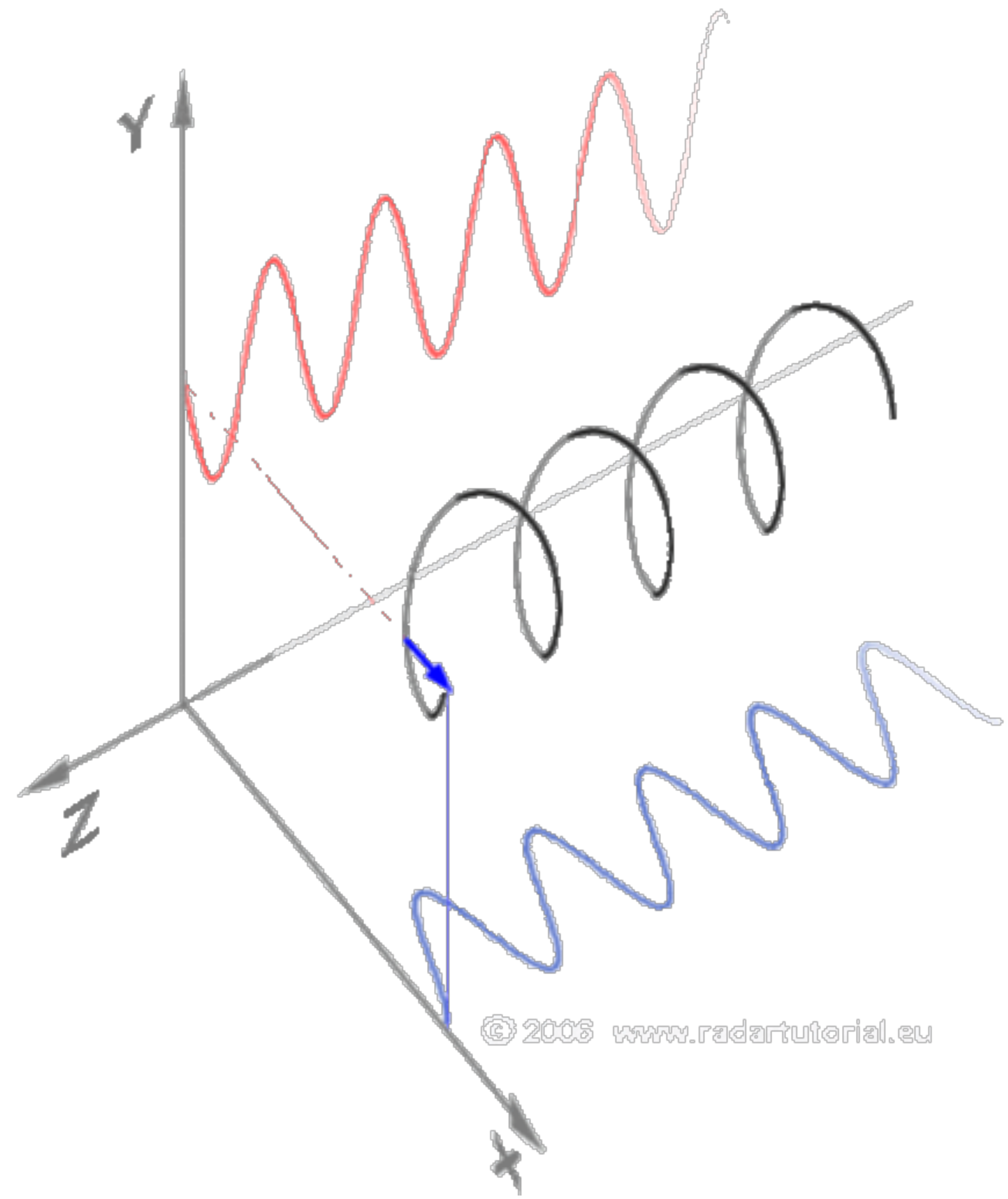
REMINDER: UNIT CIRCLE

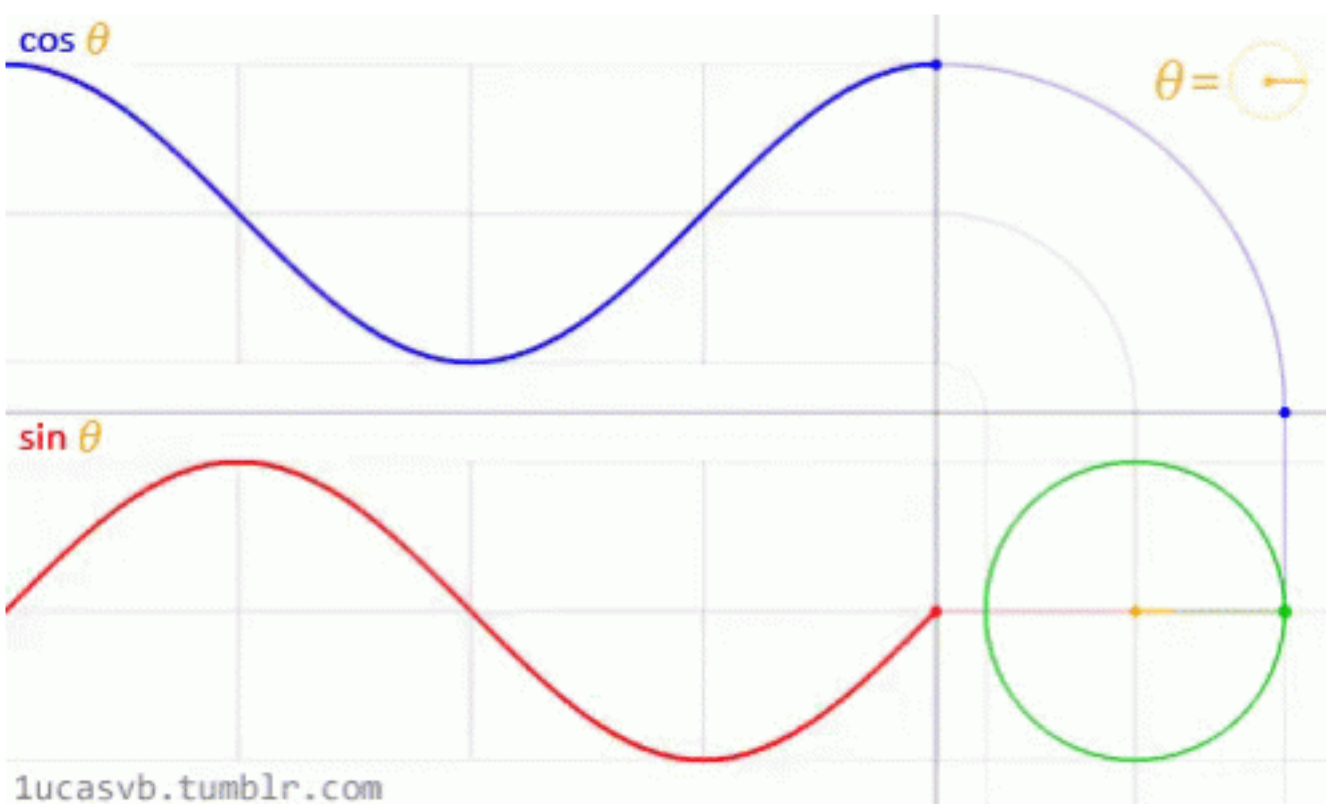


https://openlab.citytech.cuny.edu/ganguli-mat1375-fall2015/files/2015/11/186px-Unit_circle.svg_.png

REMINDER: SINES AND COSINES







http://www.businessinsider.com/7-gifs-trigonometry-sine-cosine-2013-5
https://commons.wikimedia.org/wiki/File%3ACircle_cos_sin.gif

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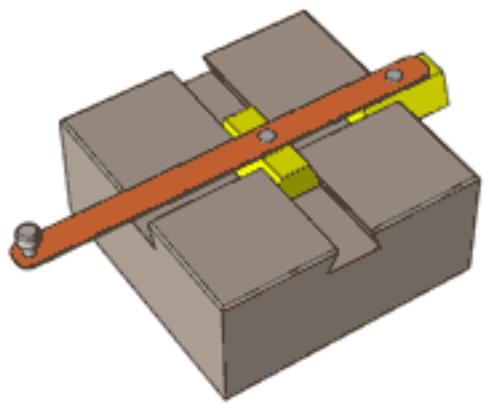

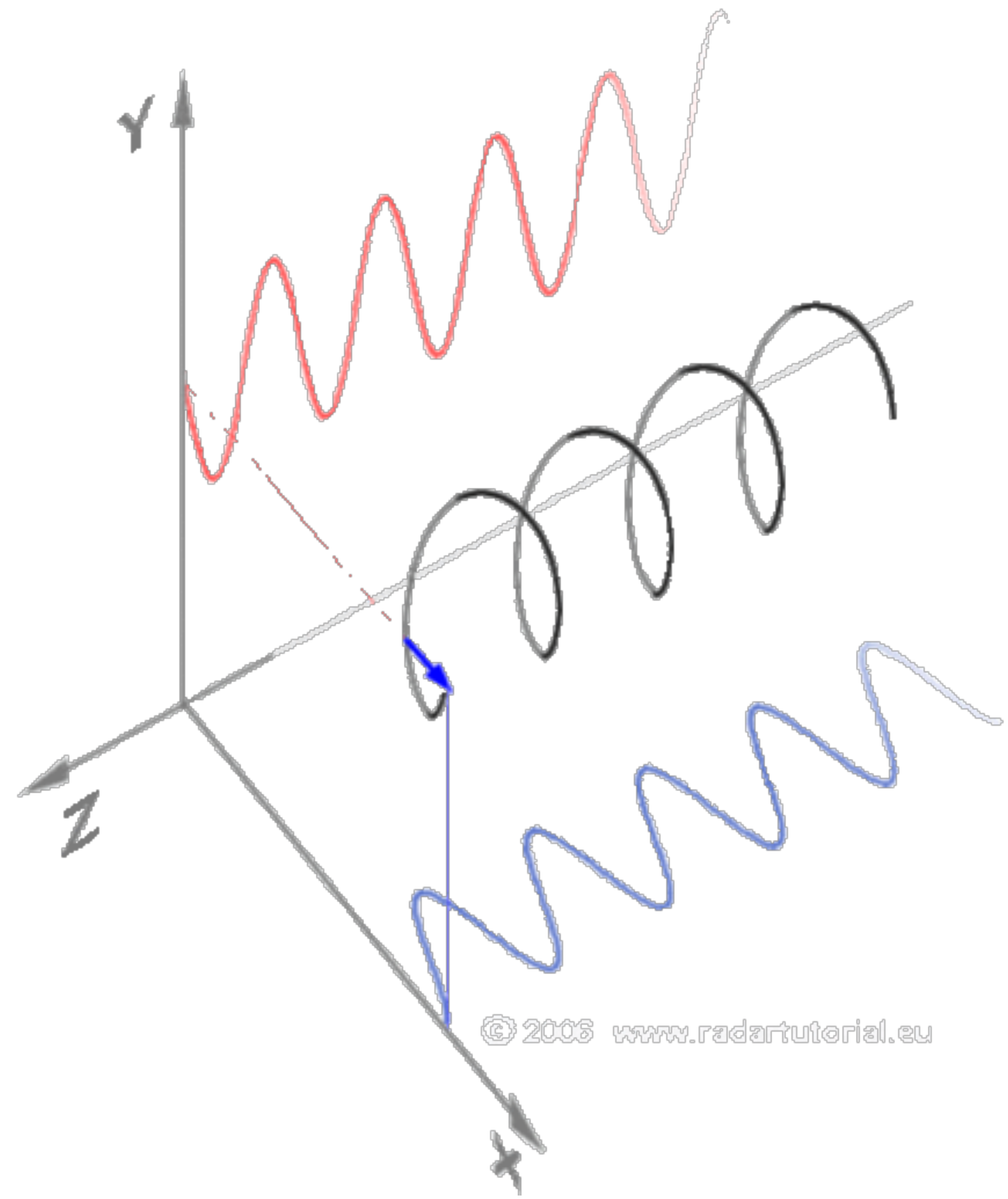

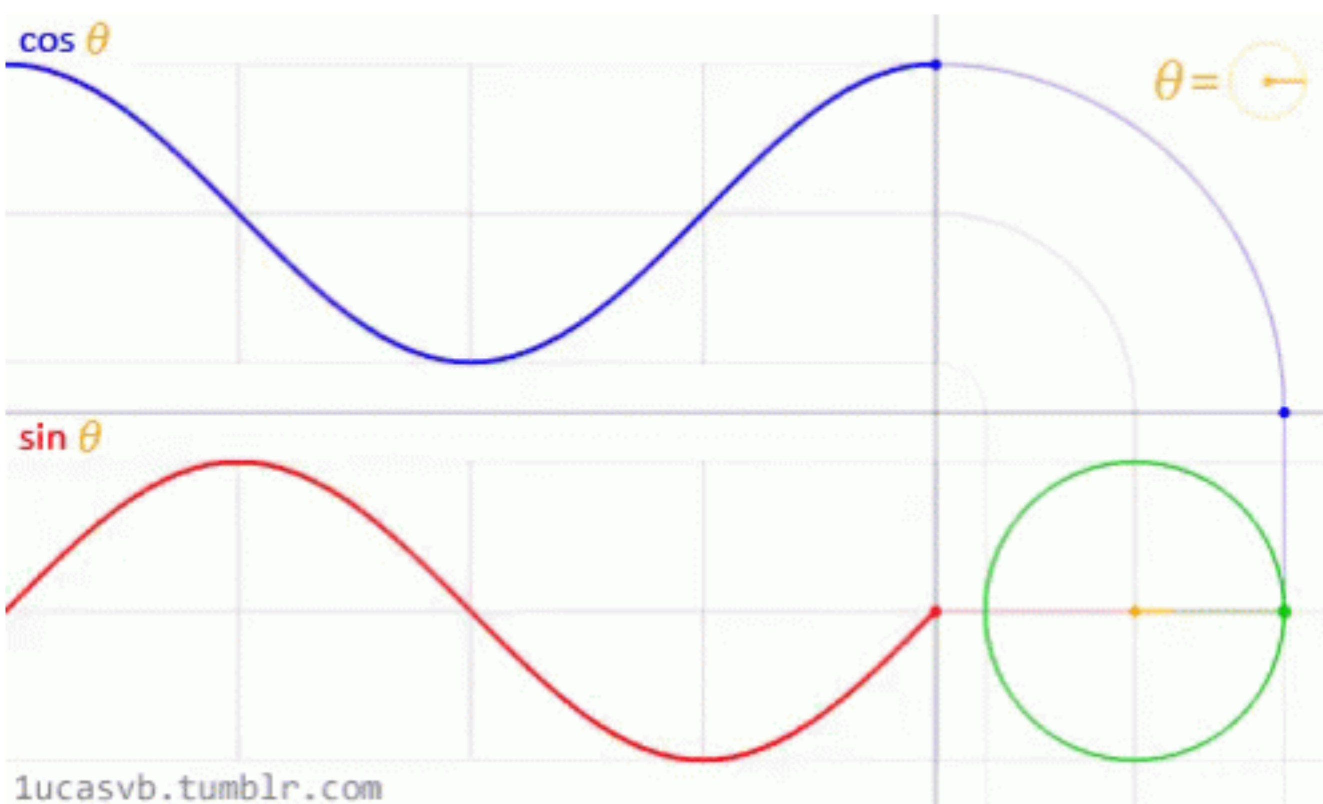

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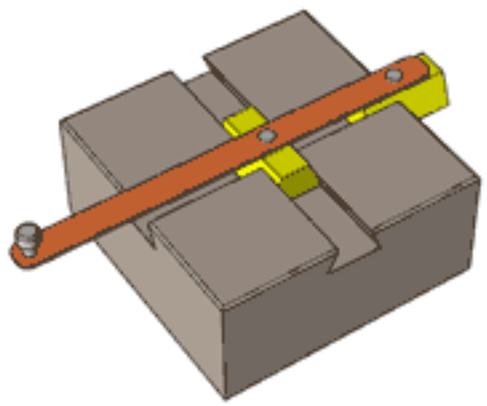

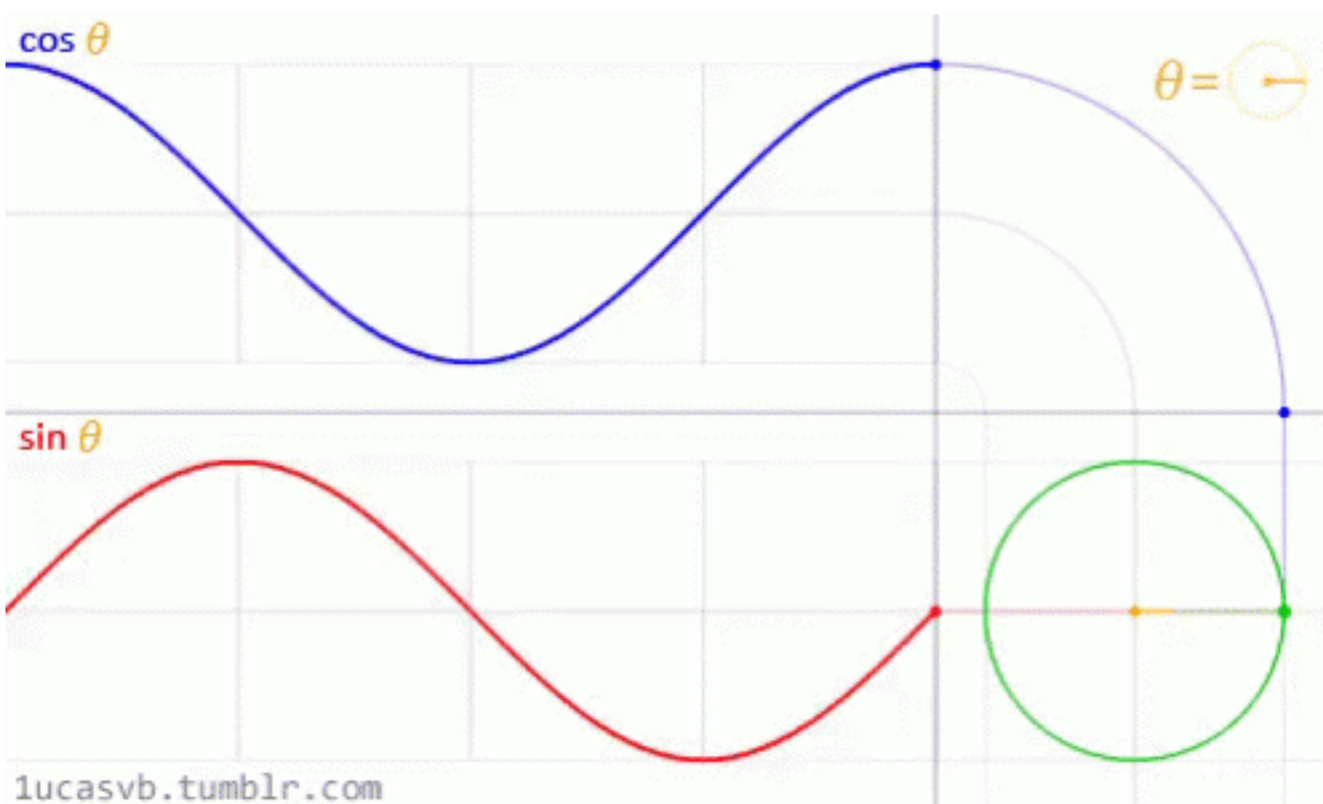

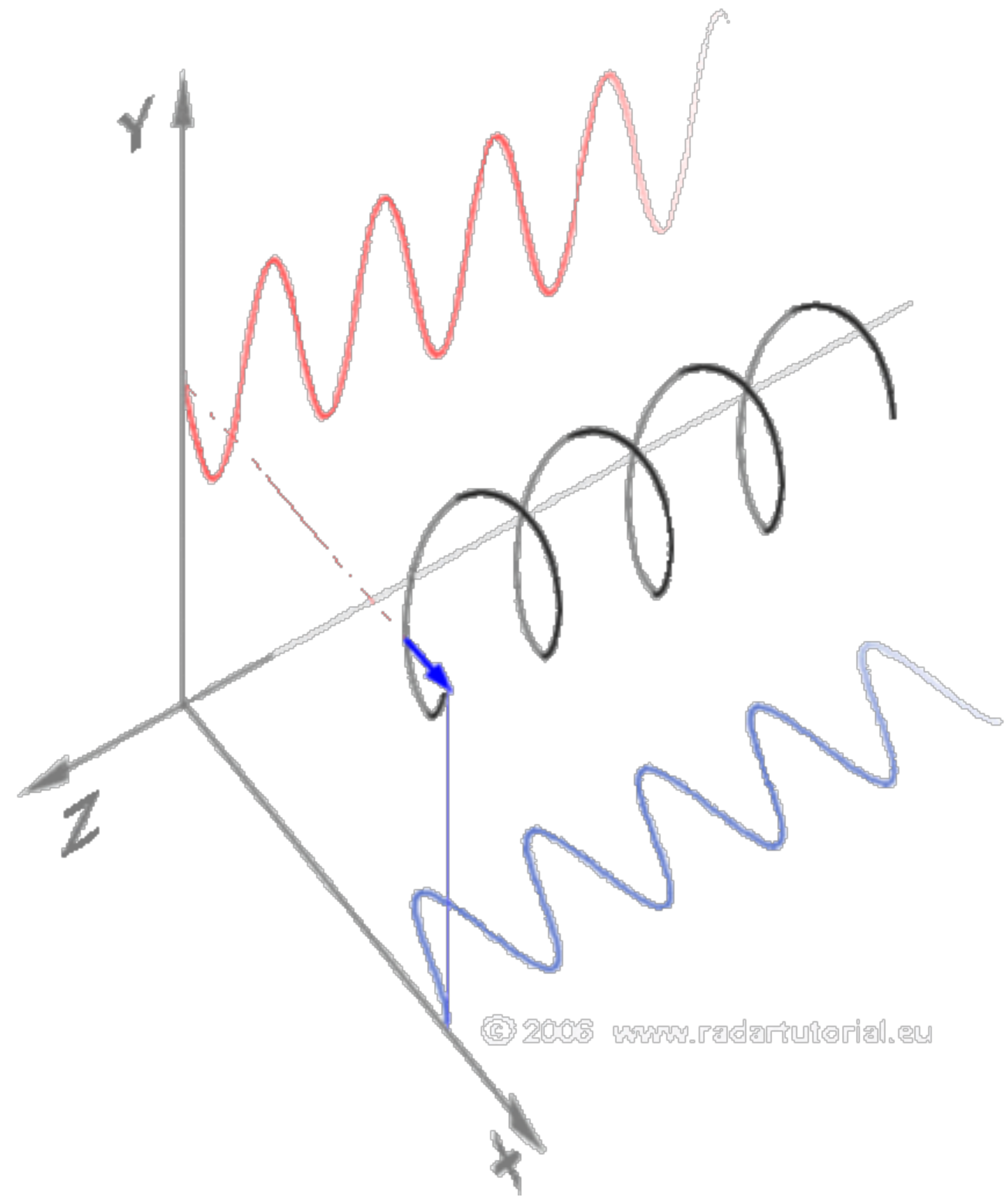

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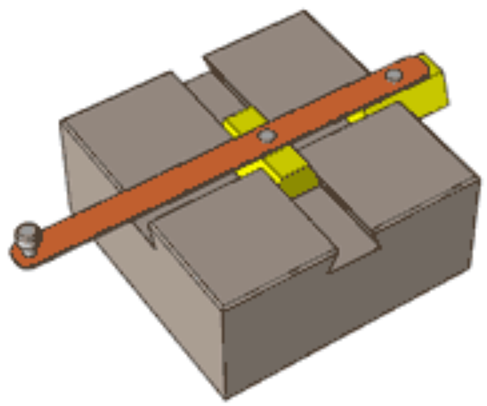

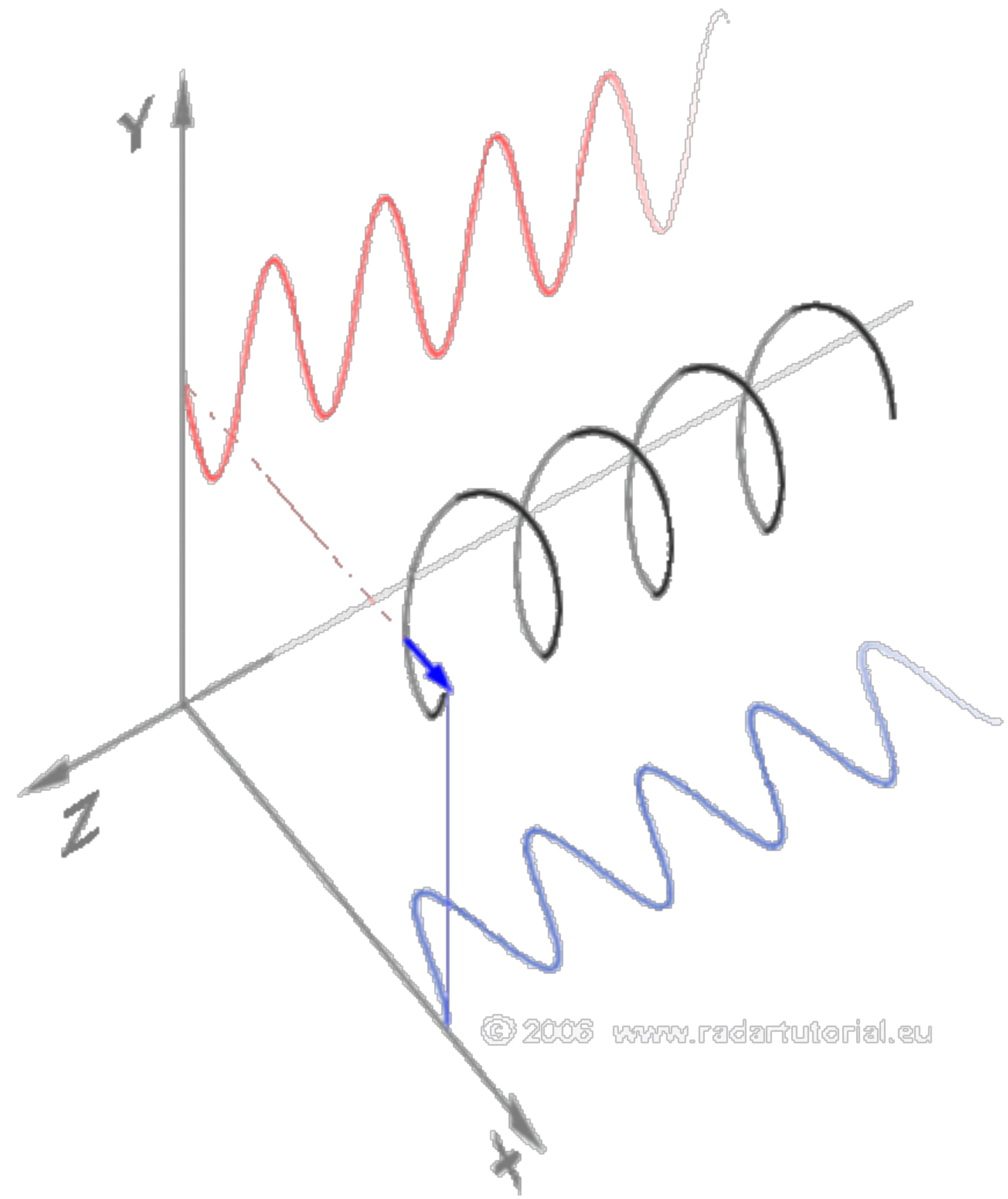

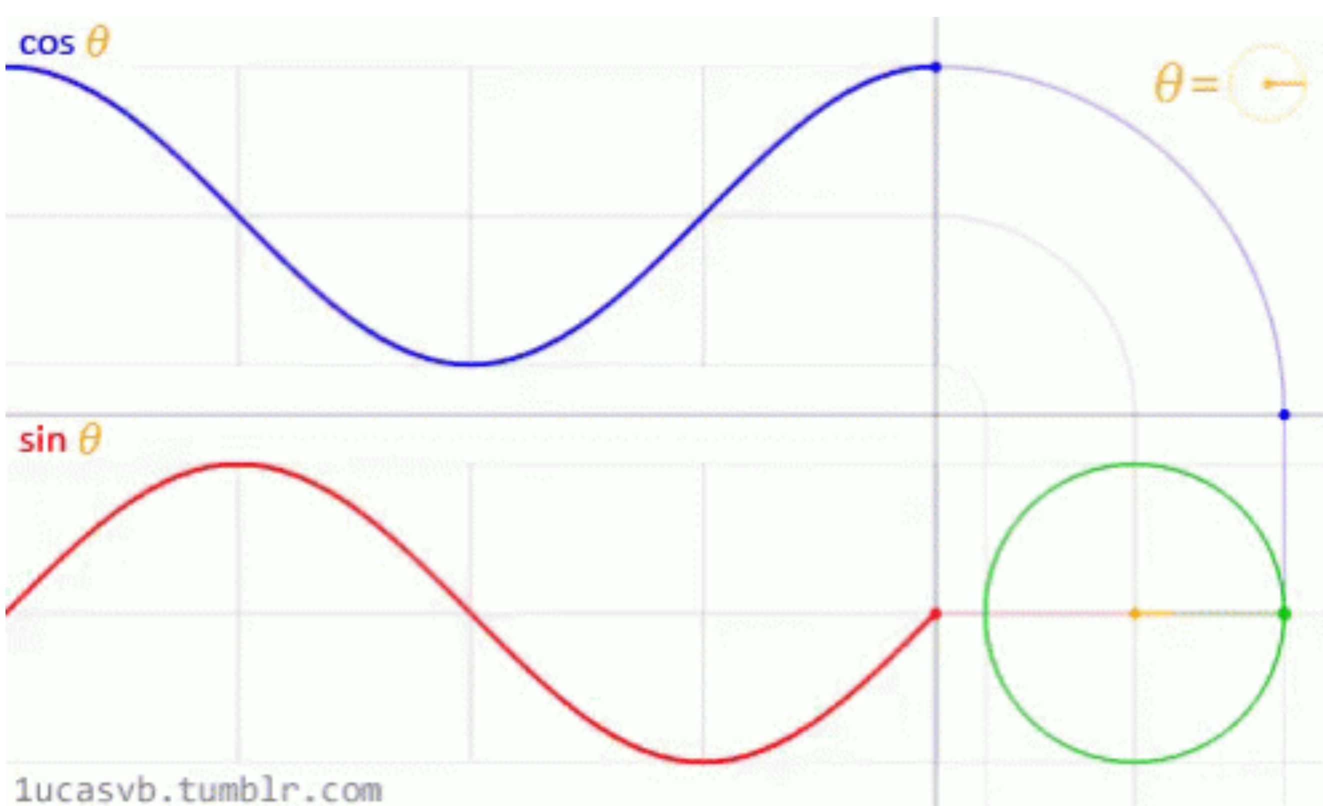

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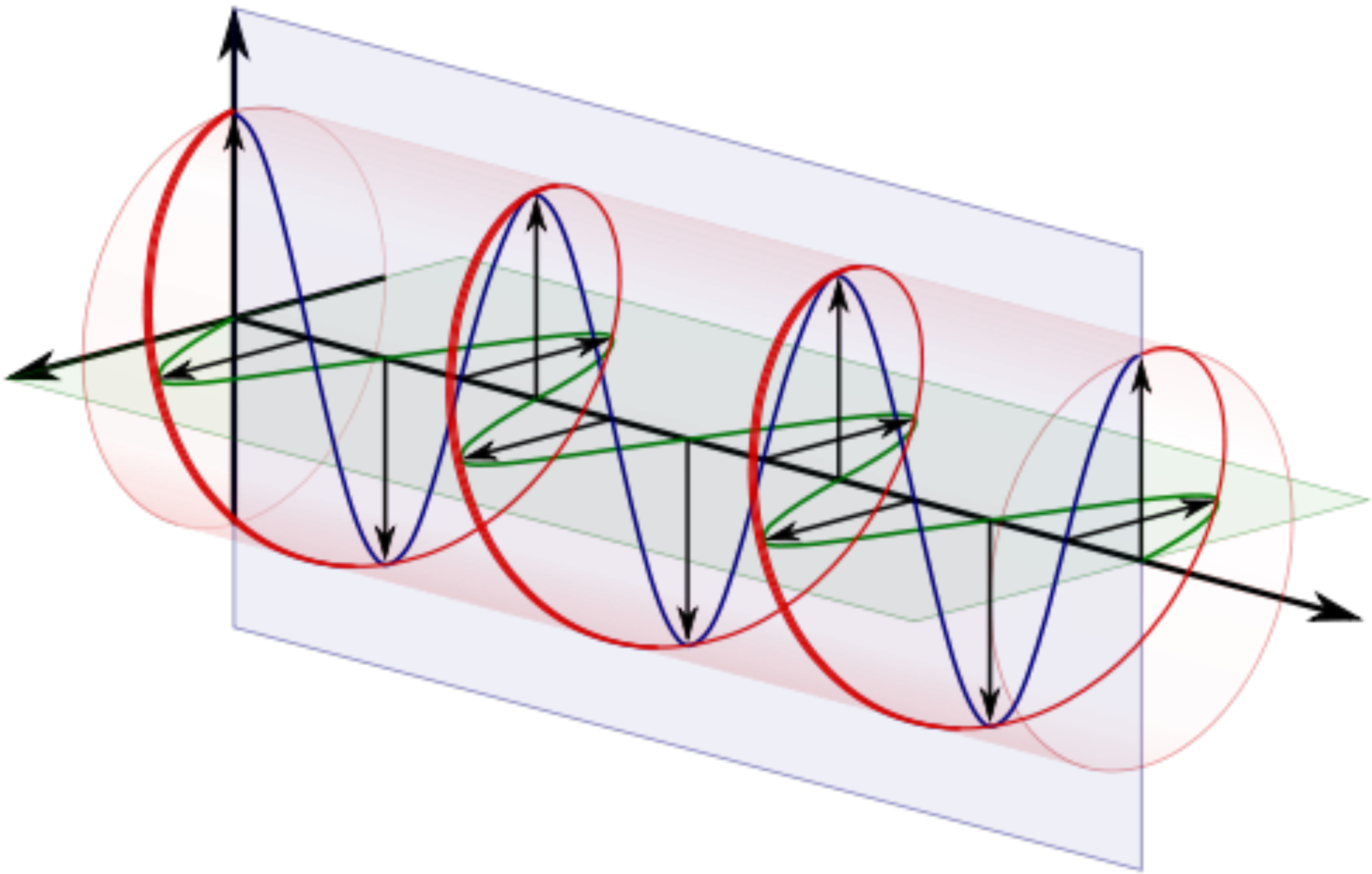
EULER'S FORMULA



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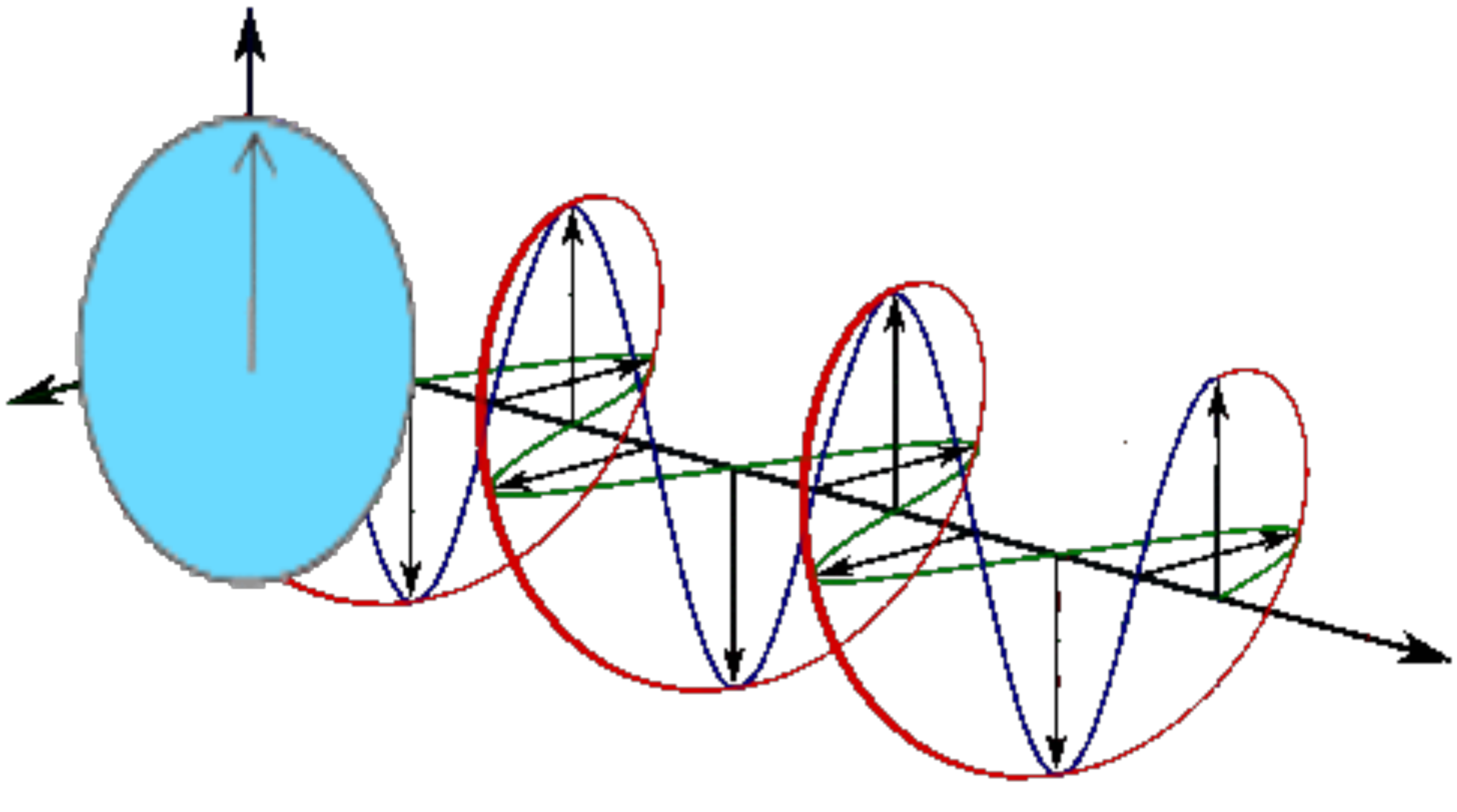
EULER'S FORMULA



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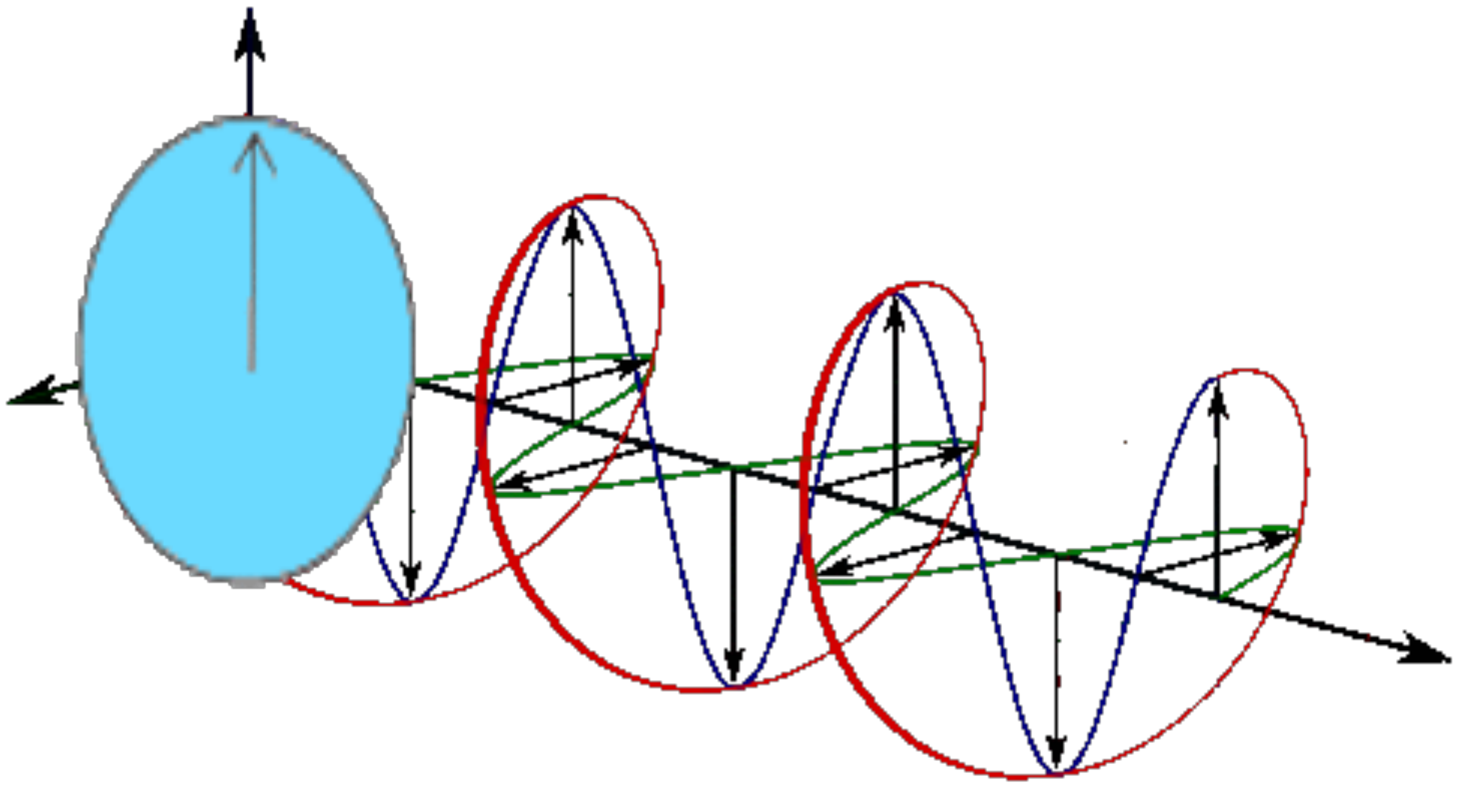
EULER'S FORMULA



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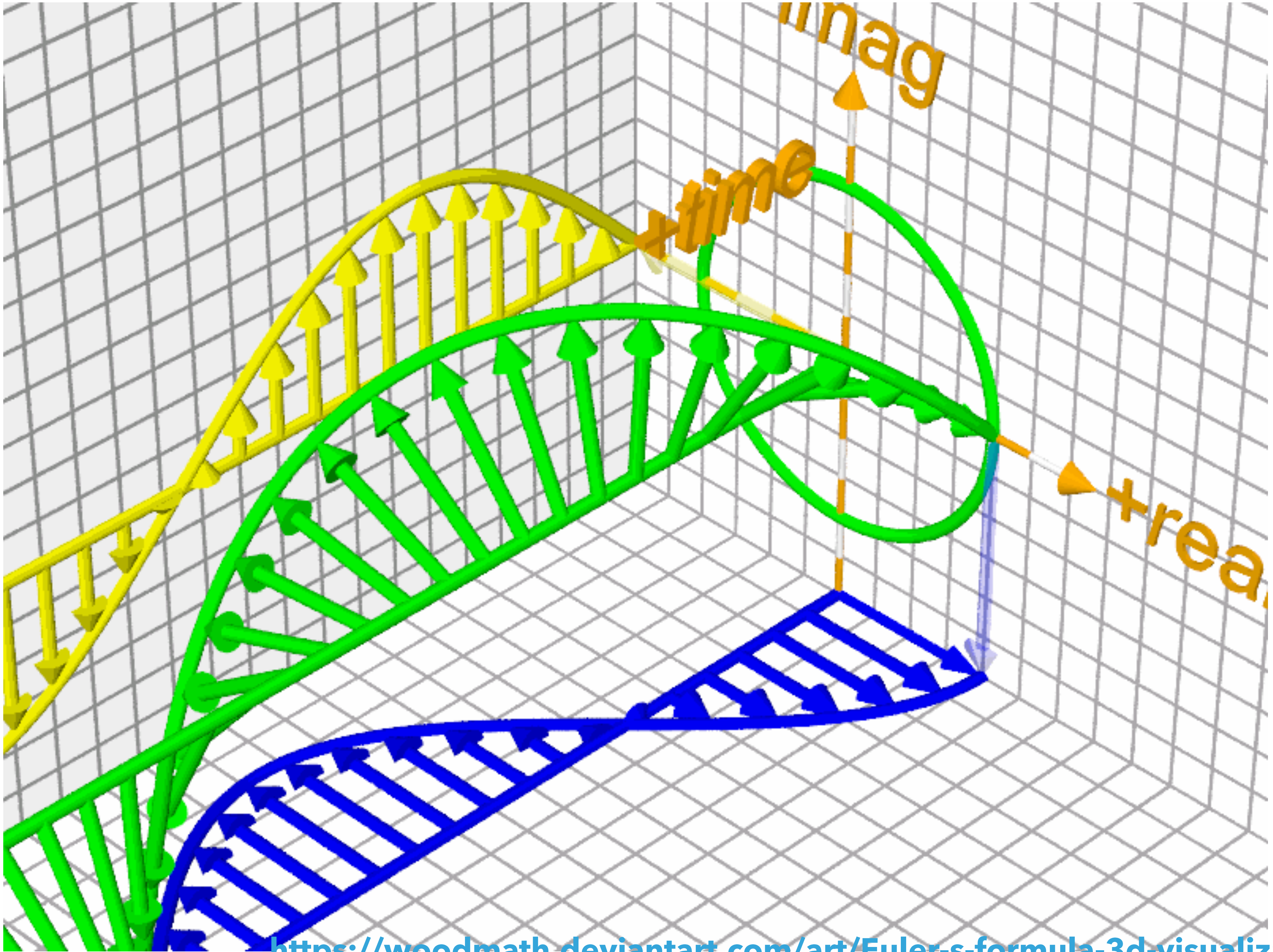
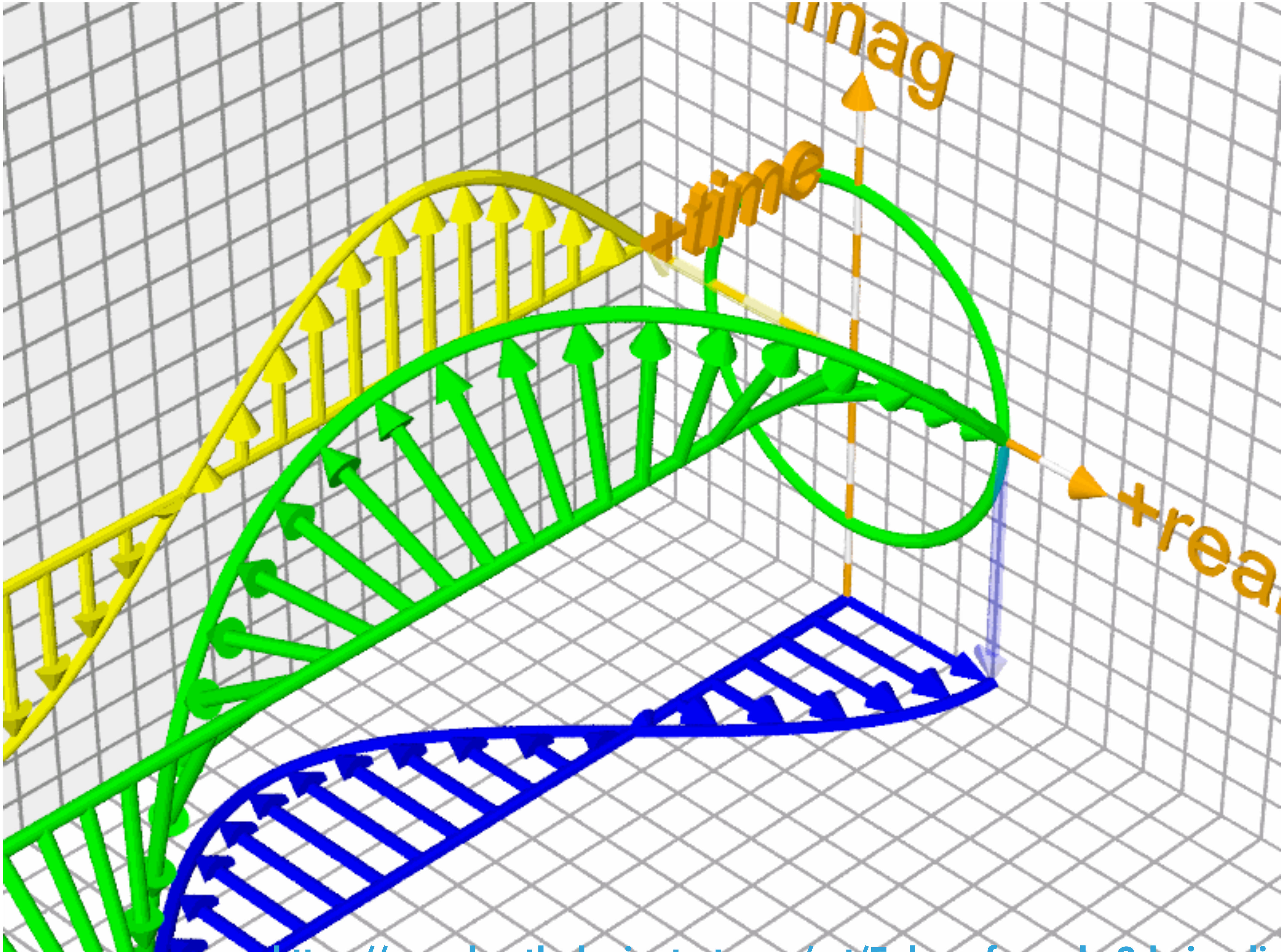


https://woodmath.deviantart.com/art/Euler-s-formula-3d-visualization-26893

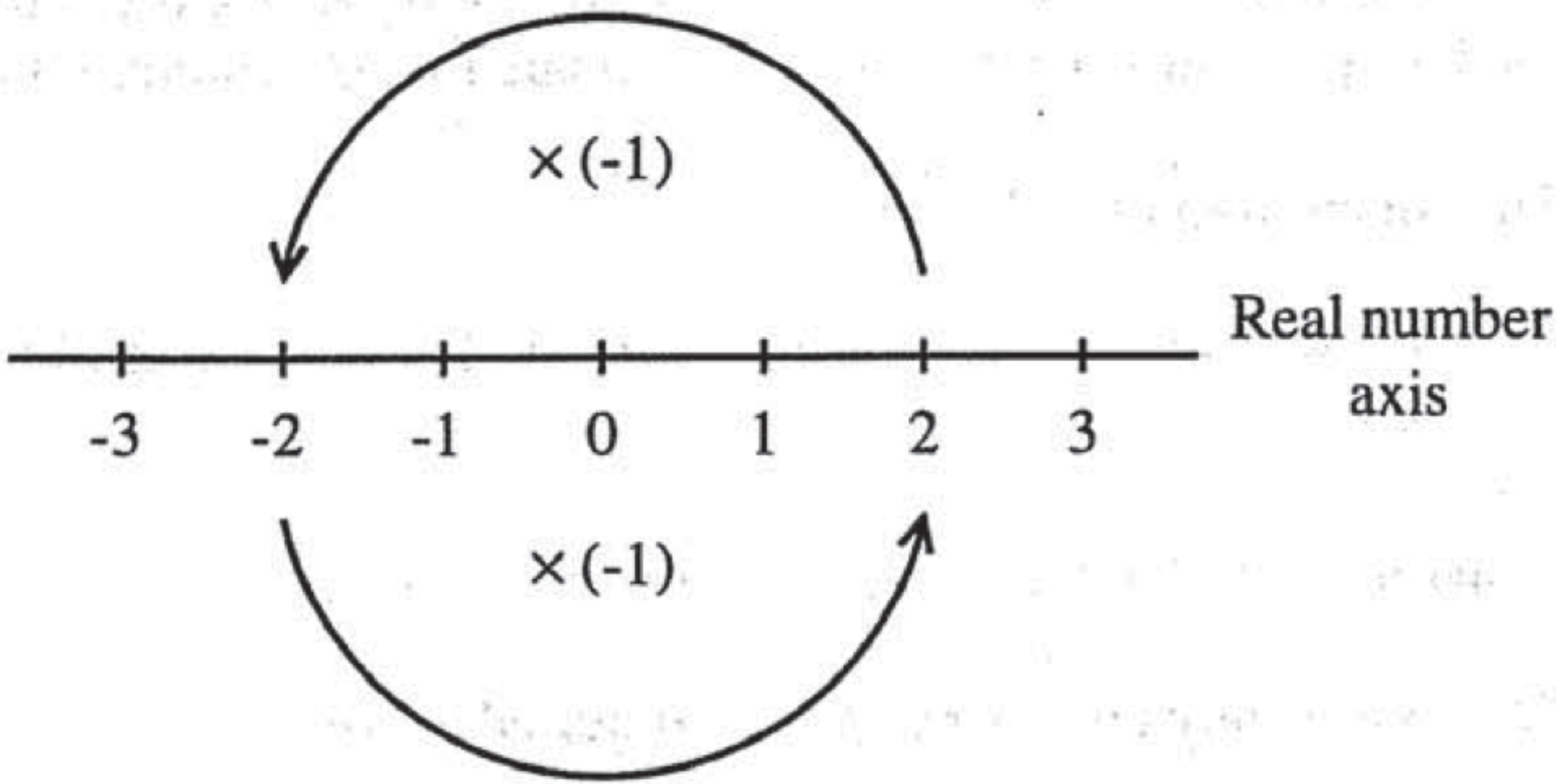
EULER'S FORMULA



https://woodmath.deviantart.com/art/Euler-s-formula-3d-visualization-268936

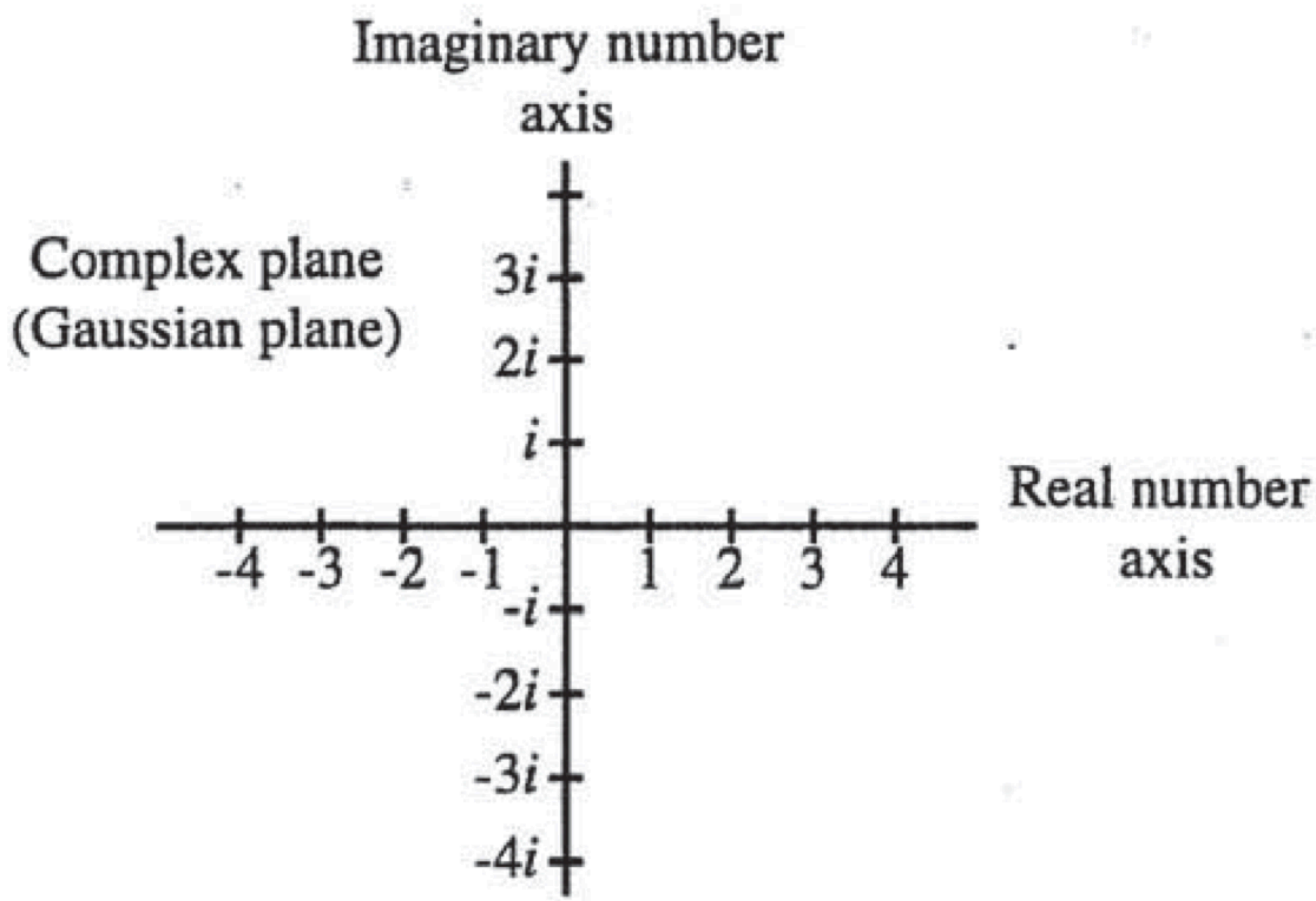
IMAGINARY NUMBERS AND ROTATION

What operation, applied twice, gets us from 1 to -1?



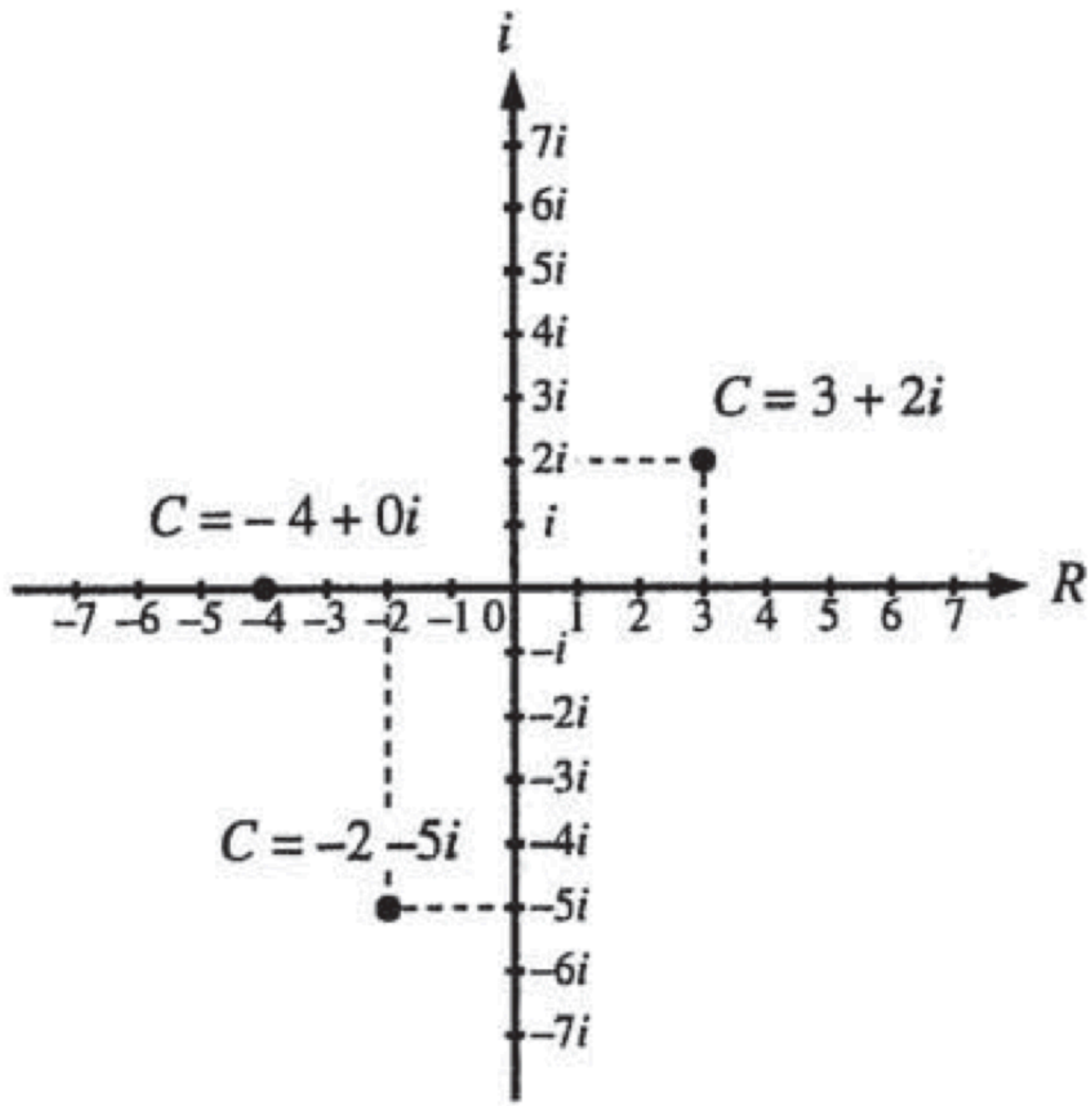
IMAGINARY NUMBERS AND ROTATION

Multiplication by *i* is a 90 degree rotation

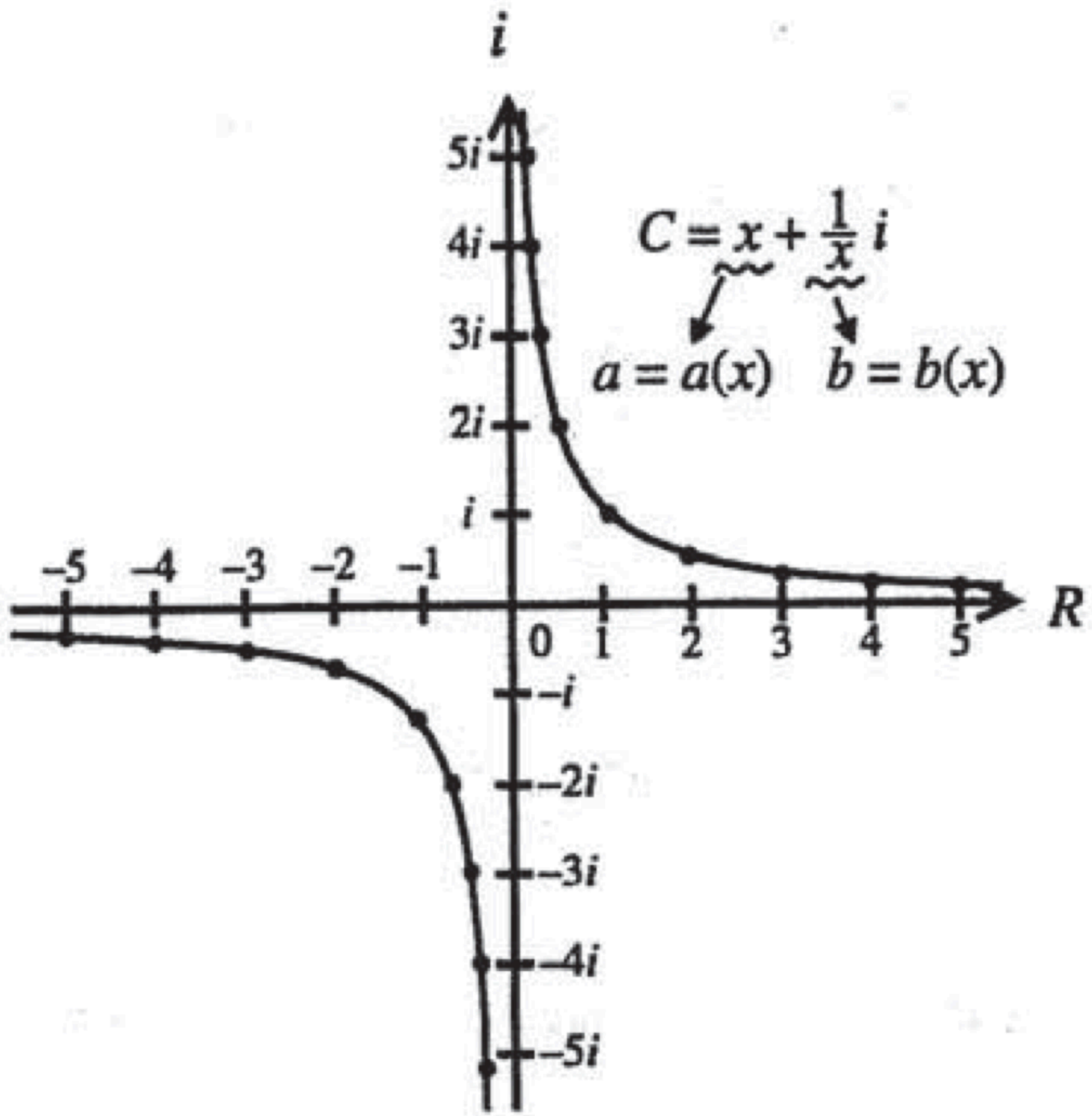


THE COMPLEX PLANE

Multiplication by *i* is a 90 degree rotation

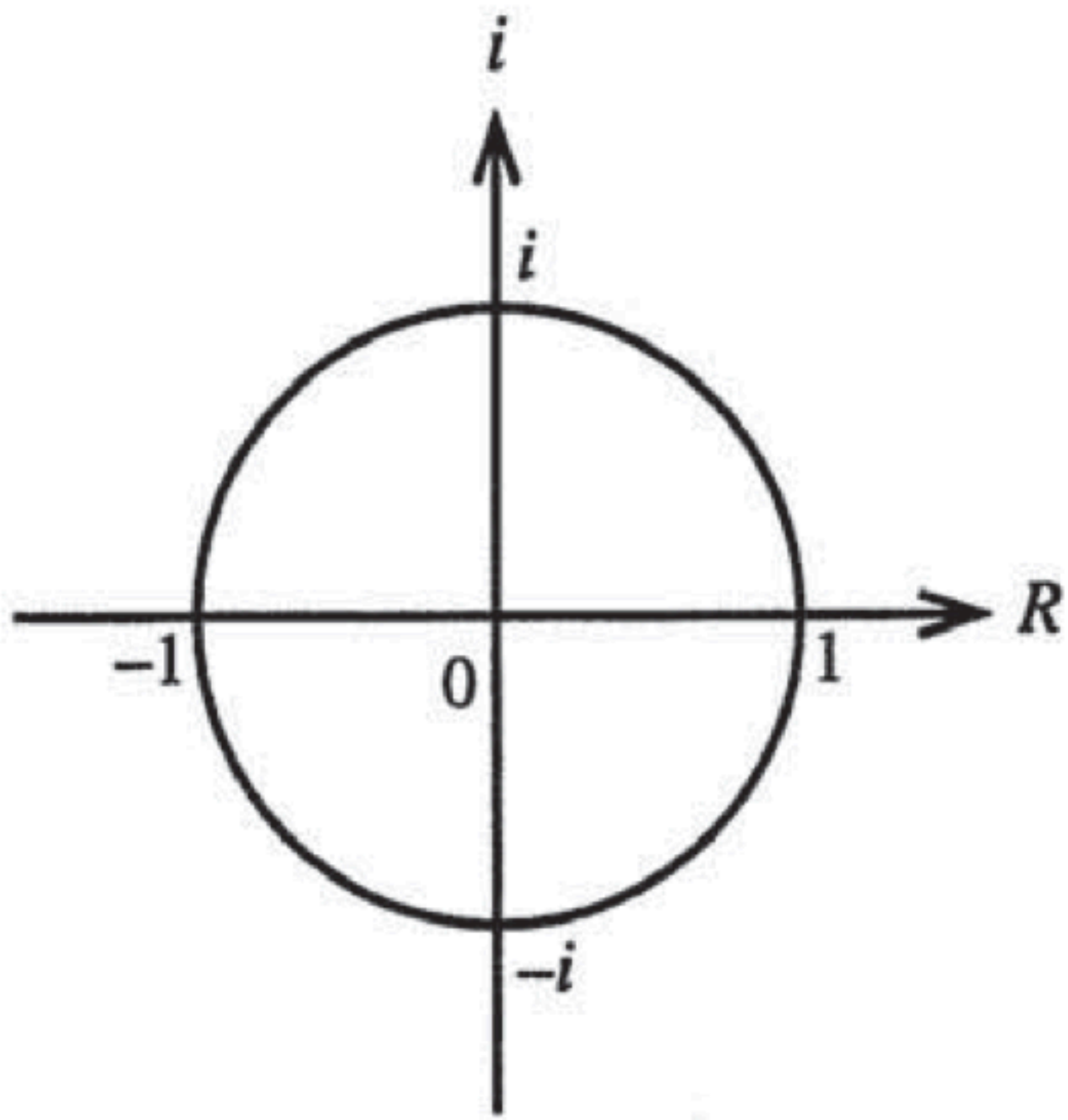


THE COMPLEX PLANE: COMPLEX FUNCTIONS



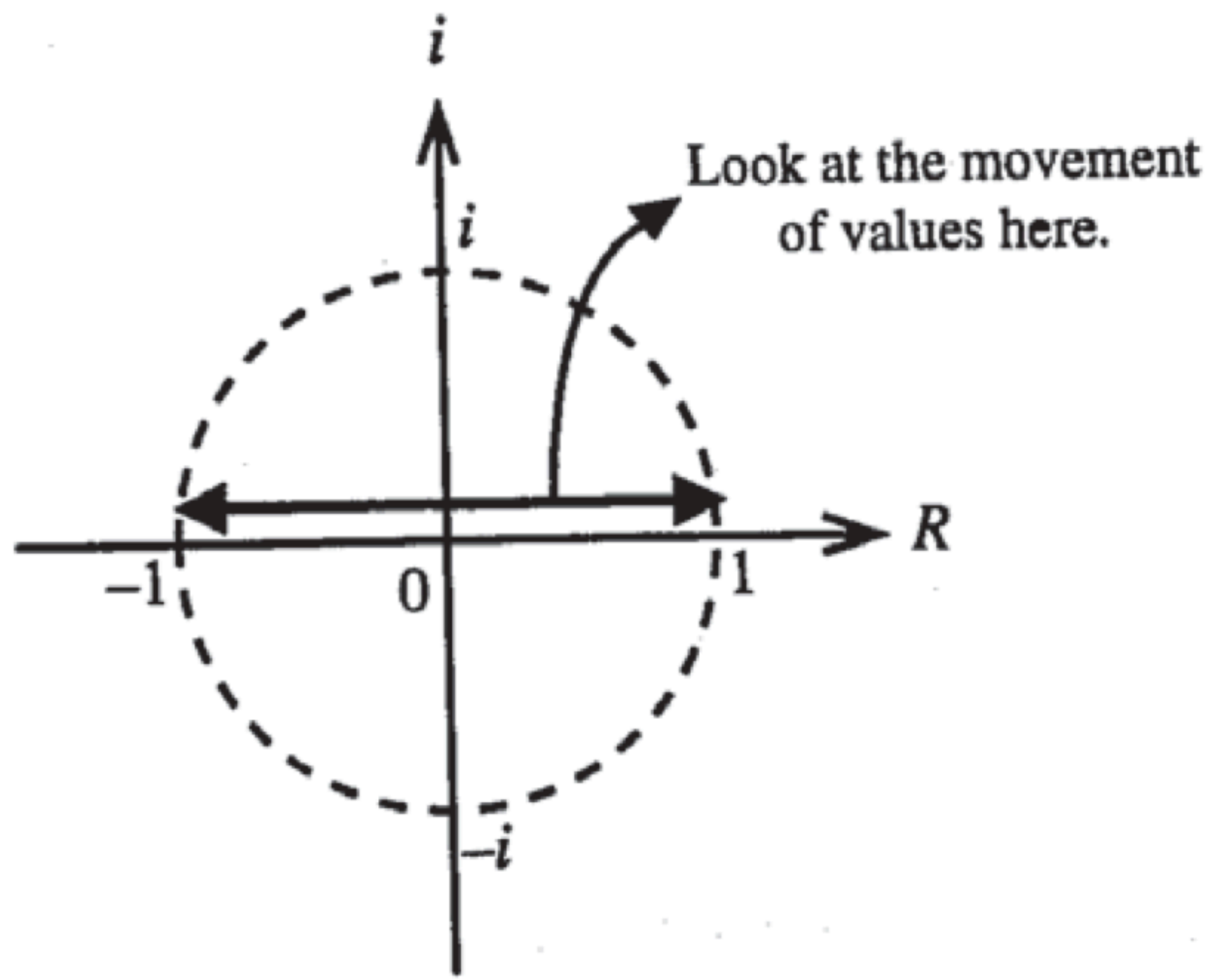
UNIT CIRCLE ON COMPLEX PLANE

$$\boxed{C(x) = a(x) + b(x)i}$$



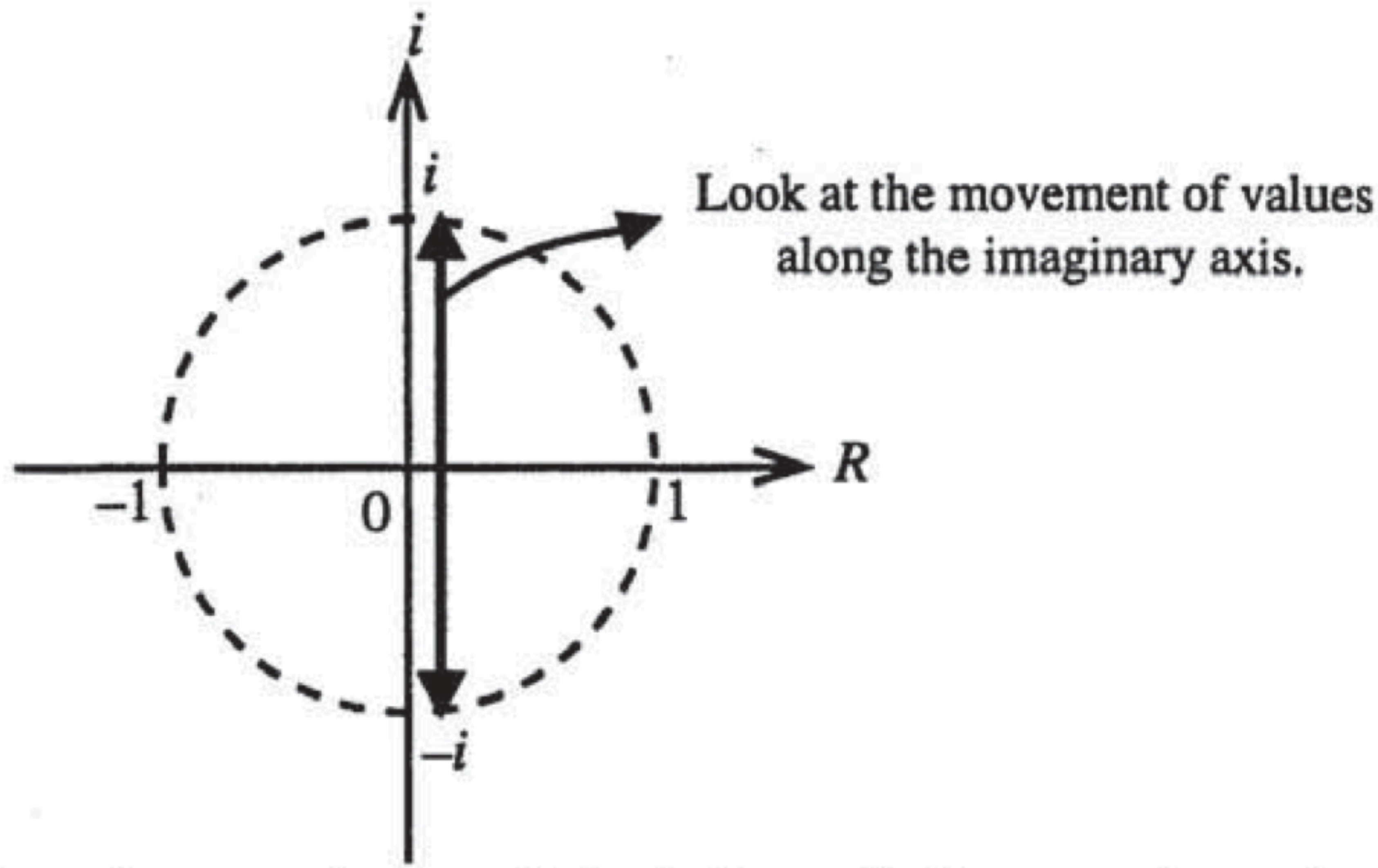
UNIT CIRCLE ON COMPLEX PLANE

$$C(x) = a(x) + b(x)i$$

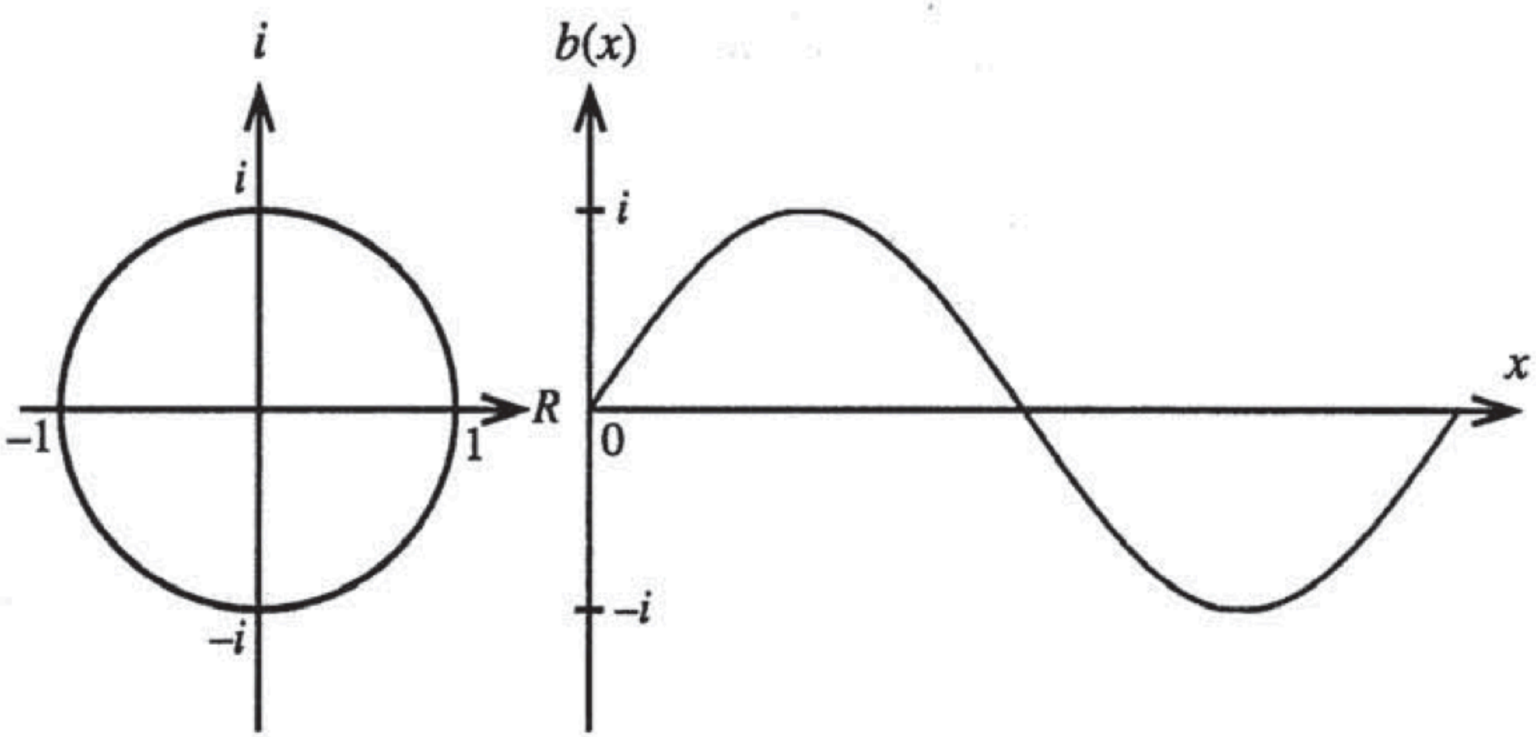


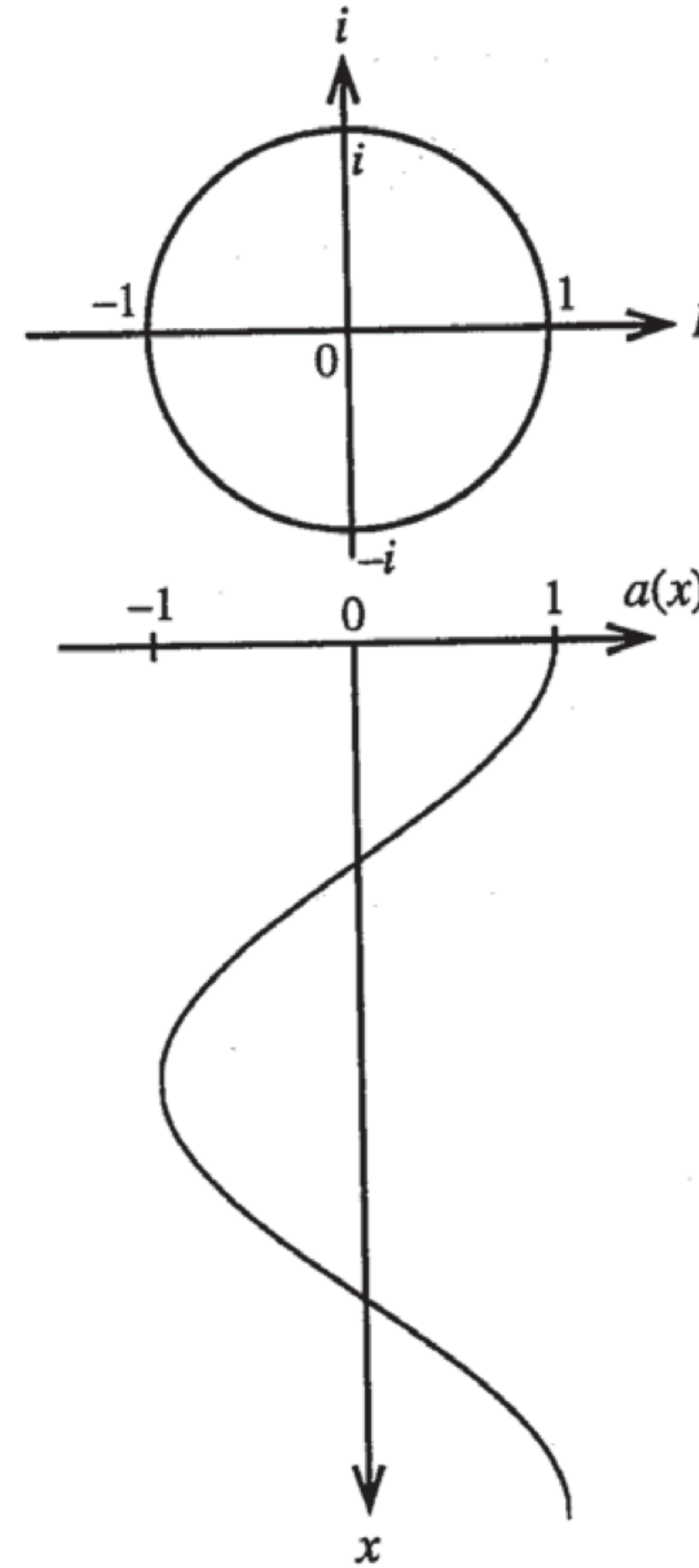
UNIT CIRCLE ON COMPLEX PLANE

$$\boxed{C(x) = a(x) + \mathbf{b(x)}i}$$



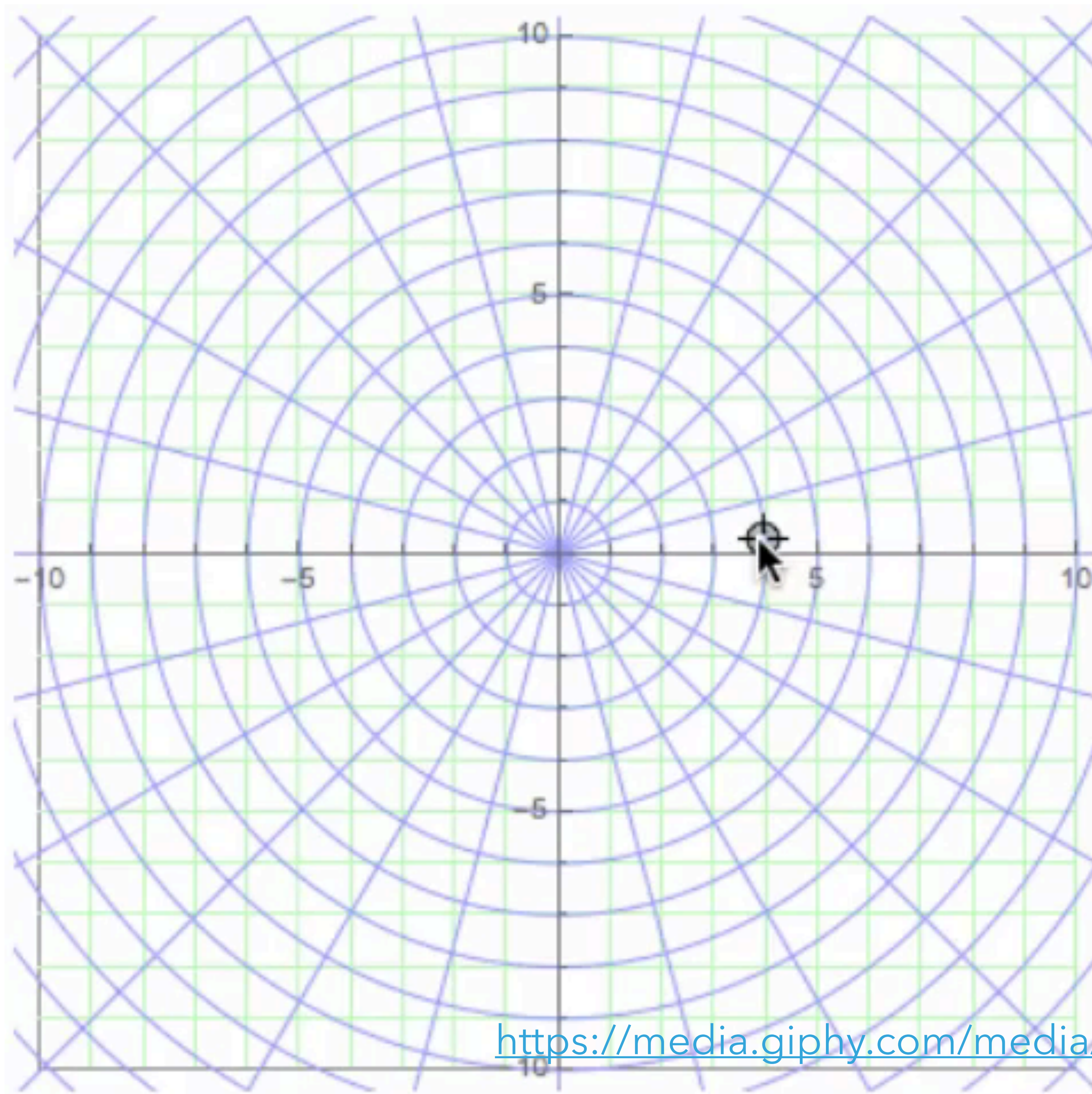
THE COMPLEX PLANE





REMINDER: POLAR COORDINATES

Polar: point $P = (r, \theta)$



https://media.giphy.com/media/sGjQtk2jLDUdy/giphy.gif

REMINDER: POLAR COORDINATES

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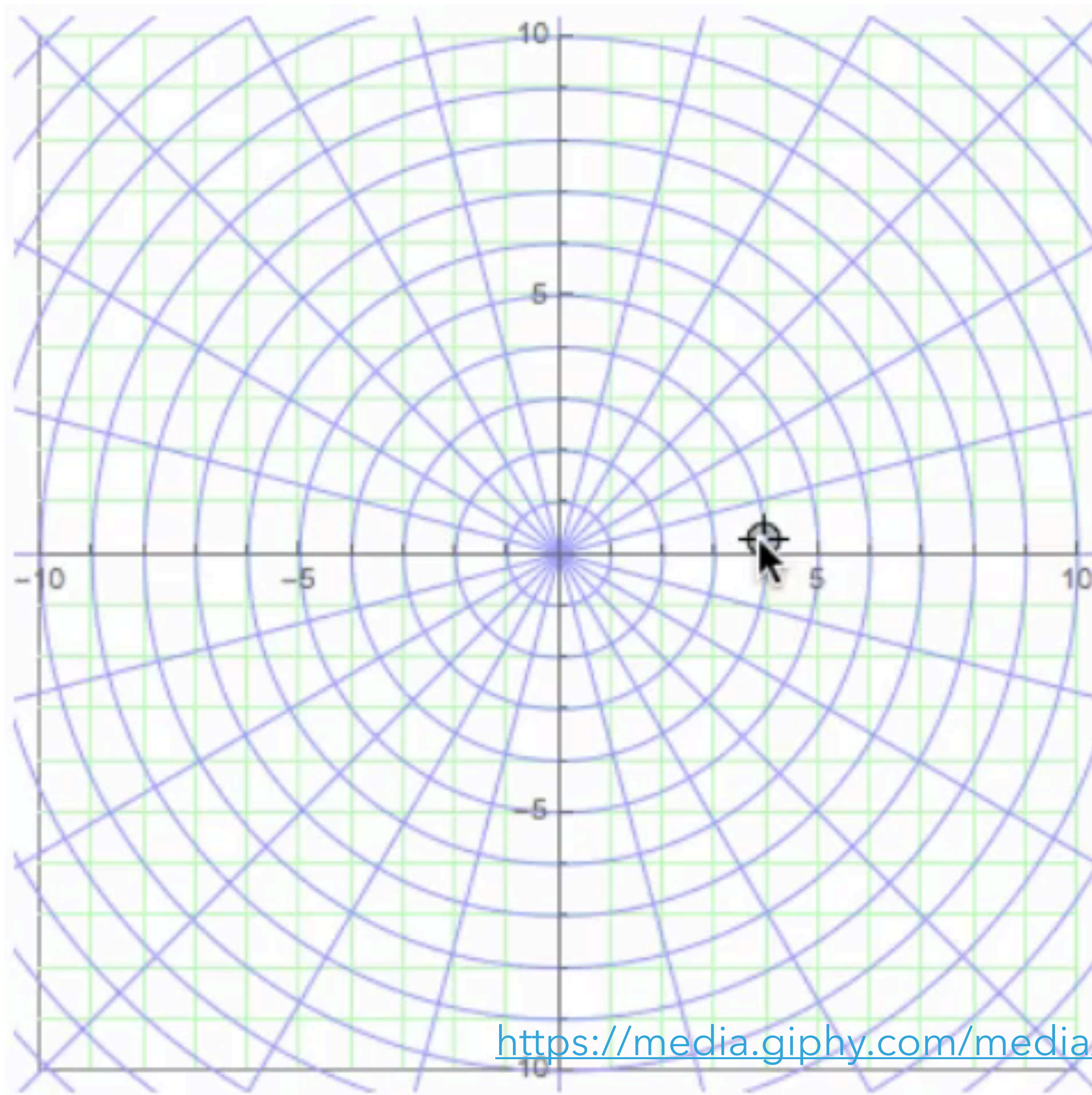
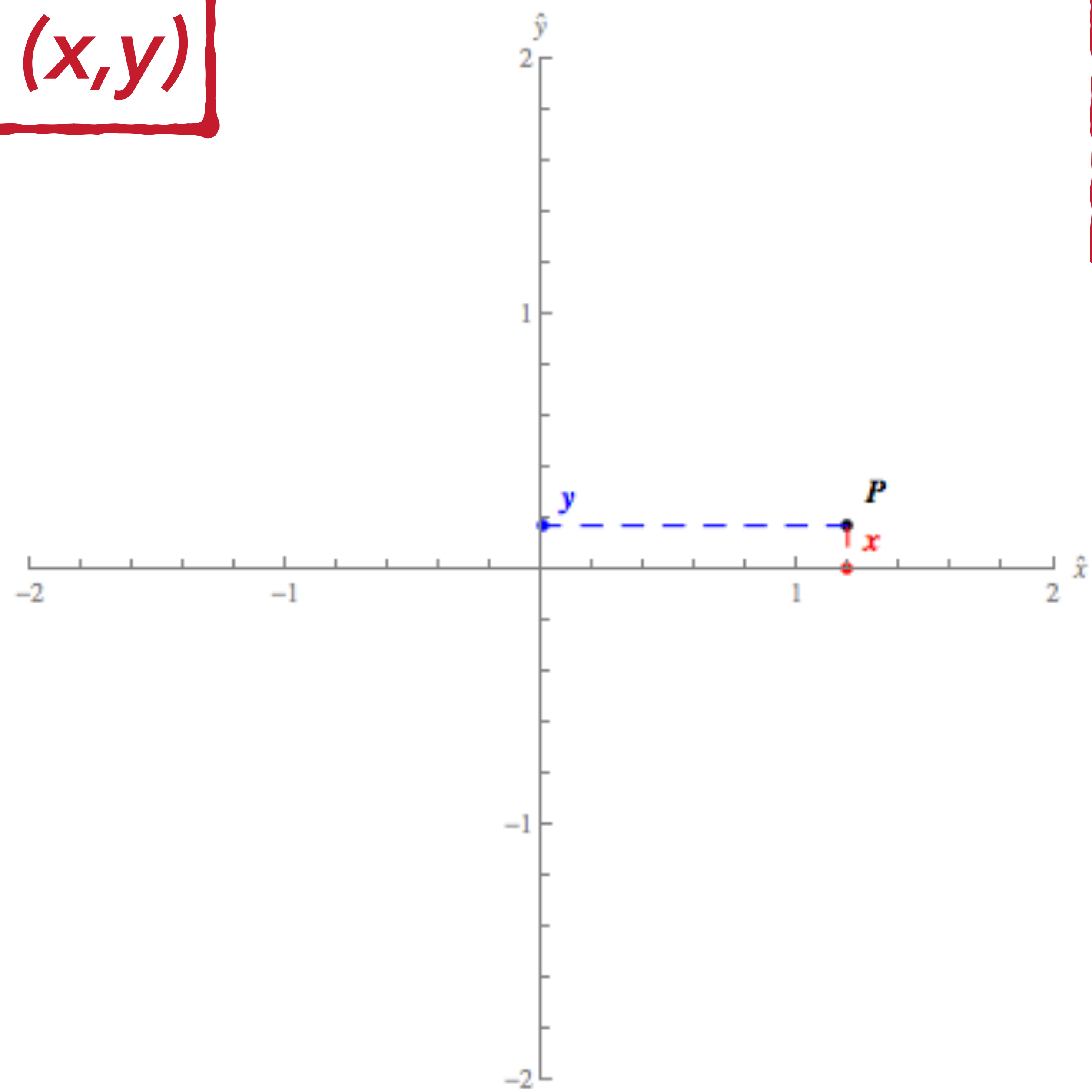


https://media.giphy.com/media/sGjQtk2jLDUdy/giphy.gif

RECTANGULAR ↔ POLAR COORDINATES

P = (*x*, *y*)
Traversal via (*x*,*y*)

P = (*r*, *θ*)
Stretch by *r*,
rotation by *θ*



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RECTANGULAR ↔ POLAR COORDINATES

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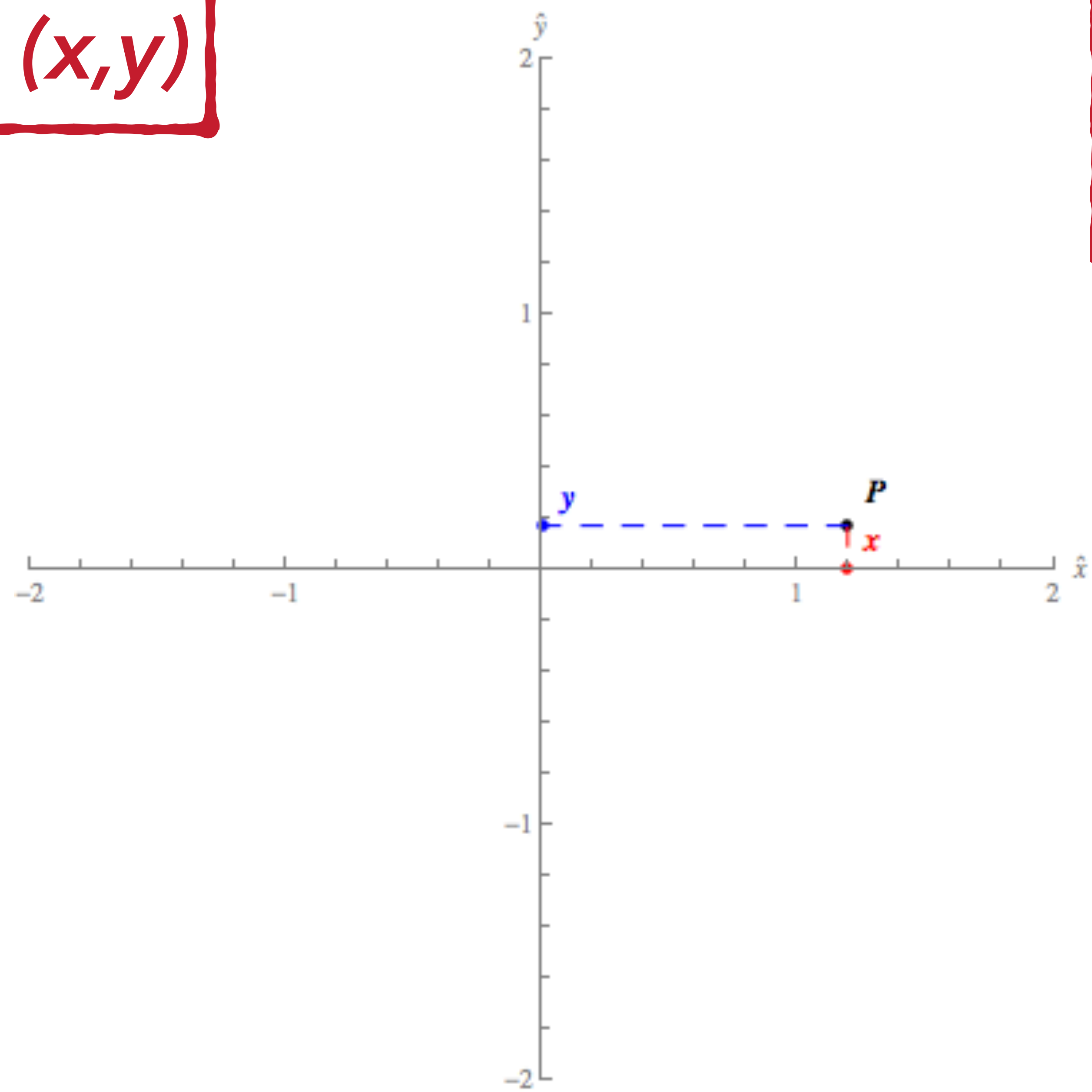
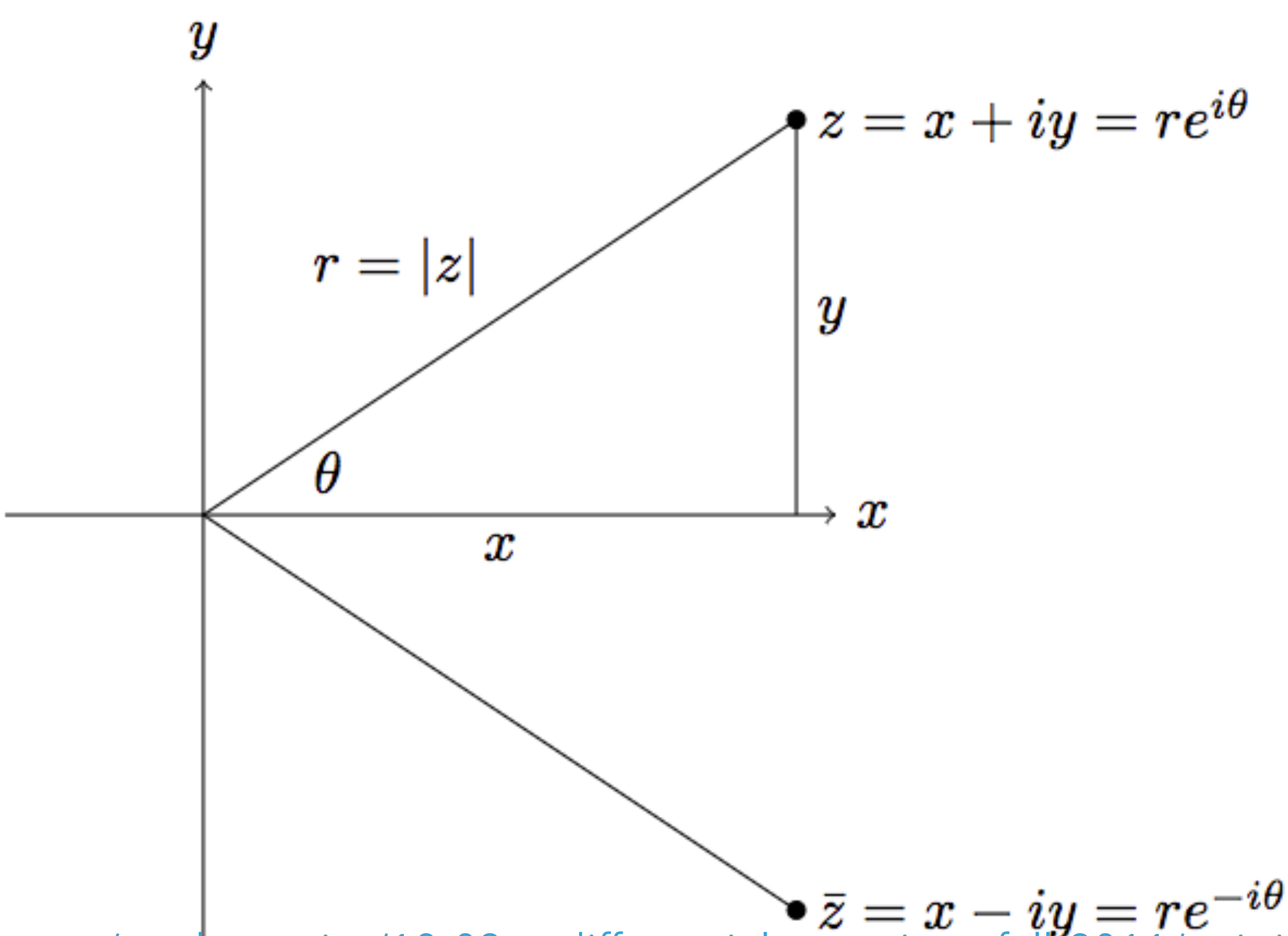


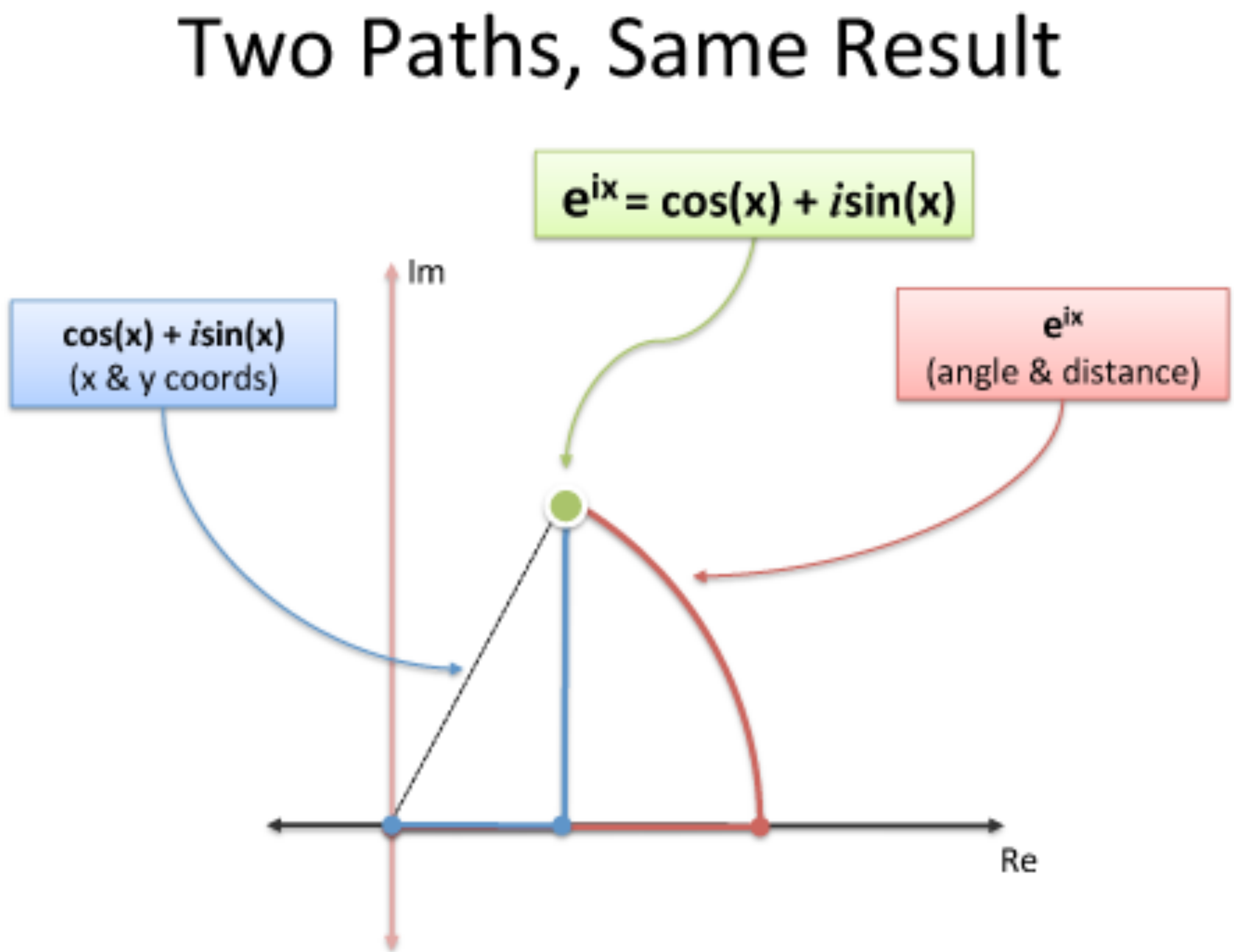
https://media.giphy.com/media/NVqZEZT4WBmuI/giphy.gif

GEOMETRIC EQUIVALENCE



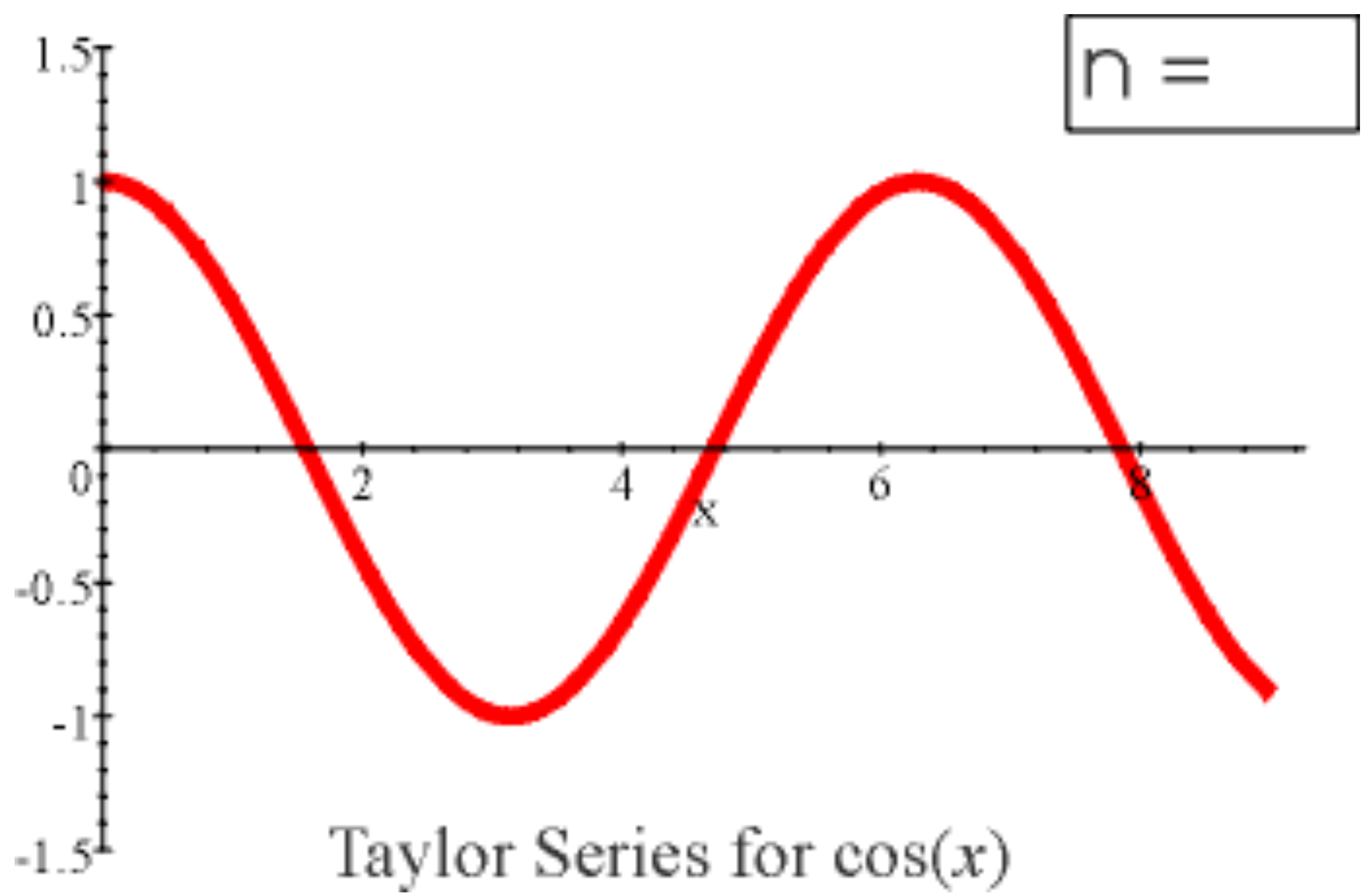
https://ocw.mit.edu/courses/mathematics/18-03sc-differential-equations-fall-2011/unit-i-first-order-differential-equations/complex-arithmetic-and-exponentials/MIT18_03SCF11_s6_5text.pdf

EULER'S FORMULA



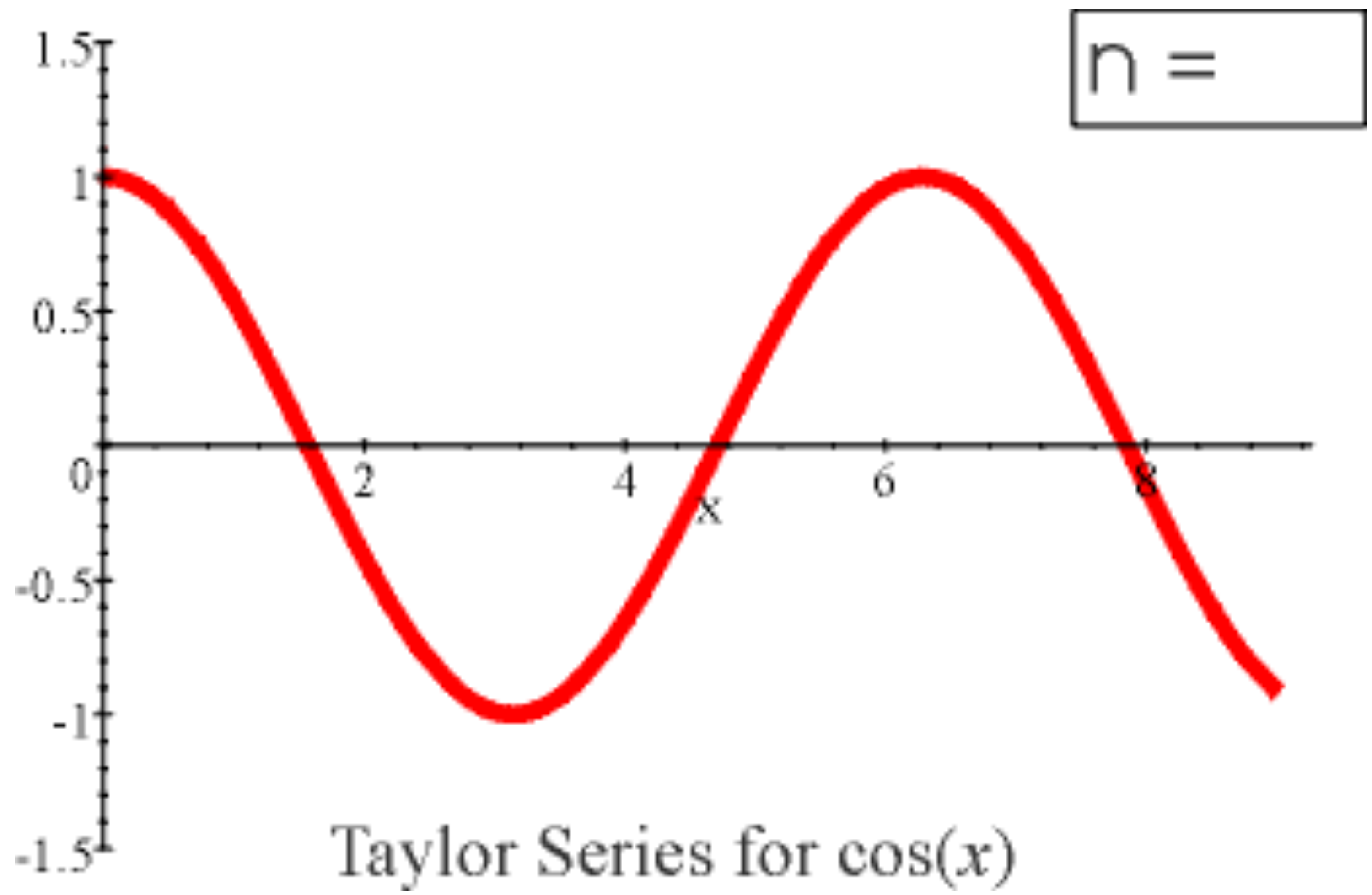
https://betterexplained.com/wp-content/uploads/euler/equal_paths.png

TAYLOR SERIES FOR COS(X)



http://www.math.tamu.edu/~dallen/cos.gif

TAYLOR SERIES FOR COS(X)



http://www.math.tamu.edu/~dallen/cos.gif

EULER'S FORMULA ALGEBRAICALLY

$$\cos(x) = 1 + 0x - \frac{1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 + 0x^5 - \frac{1}{6!}x^6 + 0x^7 + \frac{1}{8!}x^8 + \ldots$$

$$i\sin(x) = 0 + ix + 0x^2 - i\frac{1}{3!}x^3 + 0x^4 + i\frac{1}{5!}x^5 + 0x^6 - i\frac{1}{7!}x^7 + 0x^8 + \ldots$$

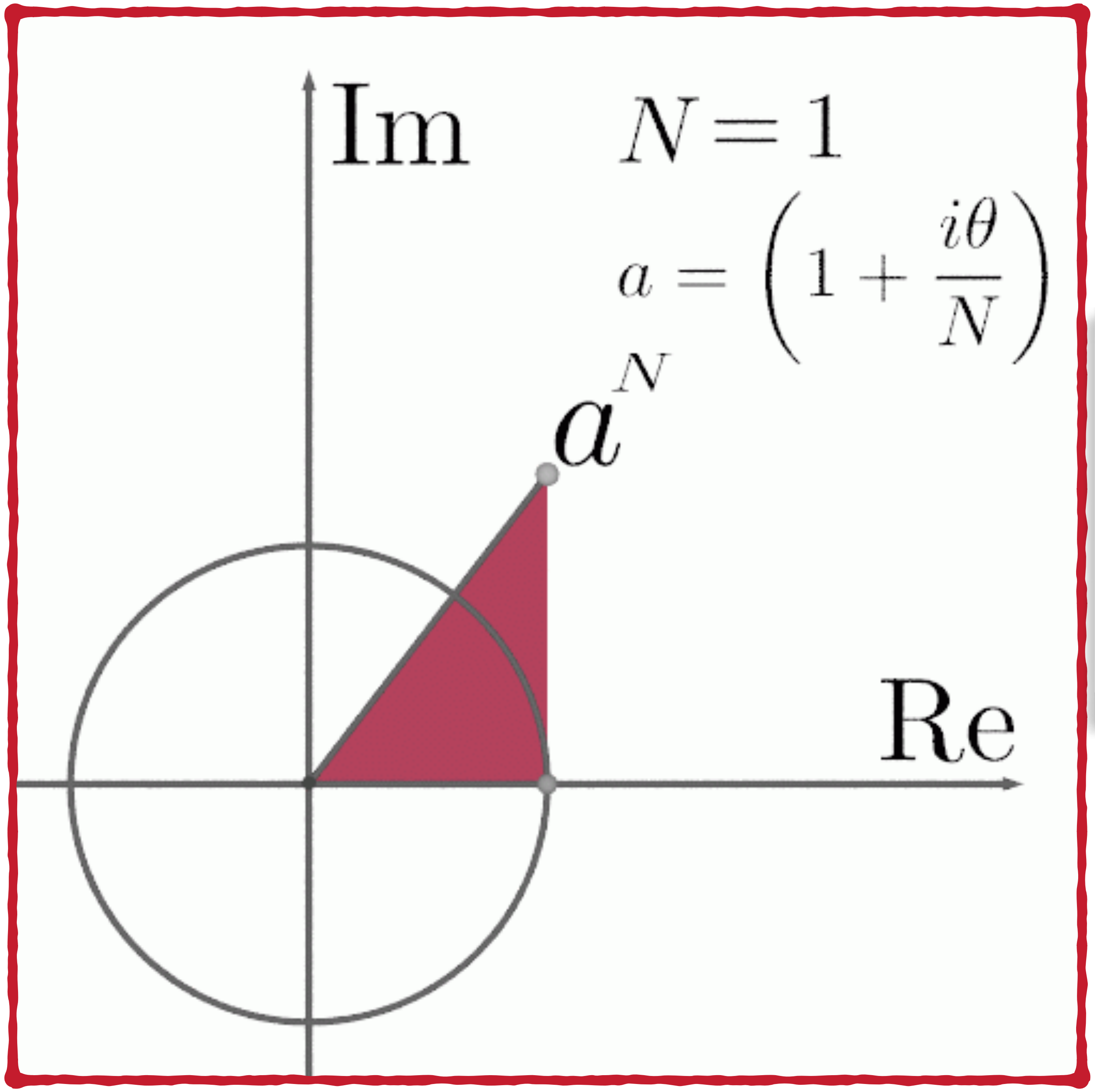
$$e^{ix} = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \ldots$$

REMINDER: WHAT IS e ?

$$\boxed{e^z = \lim_{N \to \infty} \left(1 + \frac{z}{N}\right)^N}$$

- $e = 2.718281828\ldots$
- **Special property:**
  - When you differentiate or integrate e^x you get back e^x

ANOTHER LIMIT EXPLANATION GRAPHICALLY



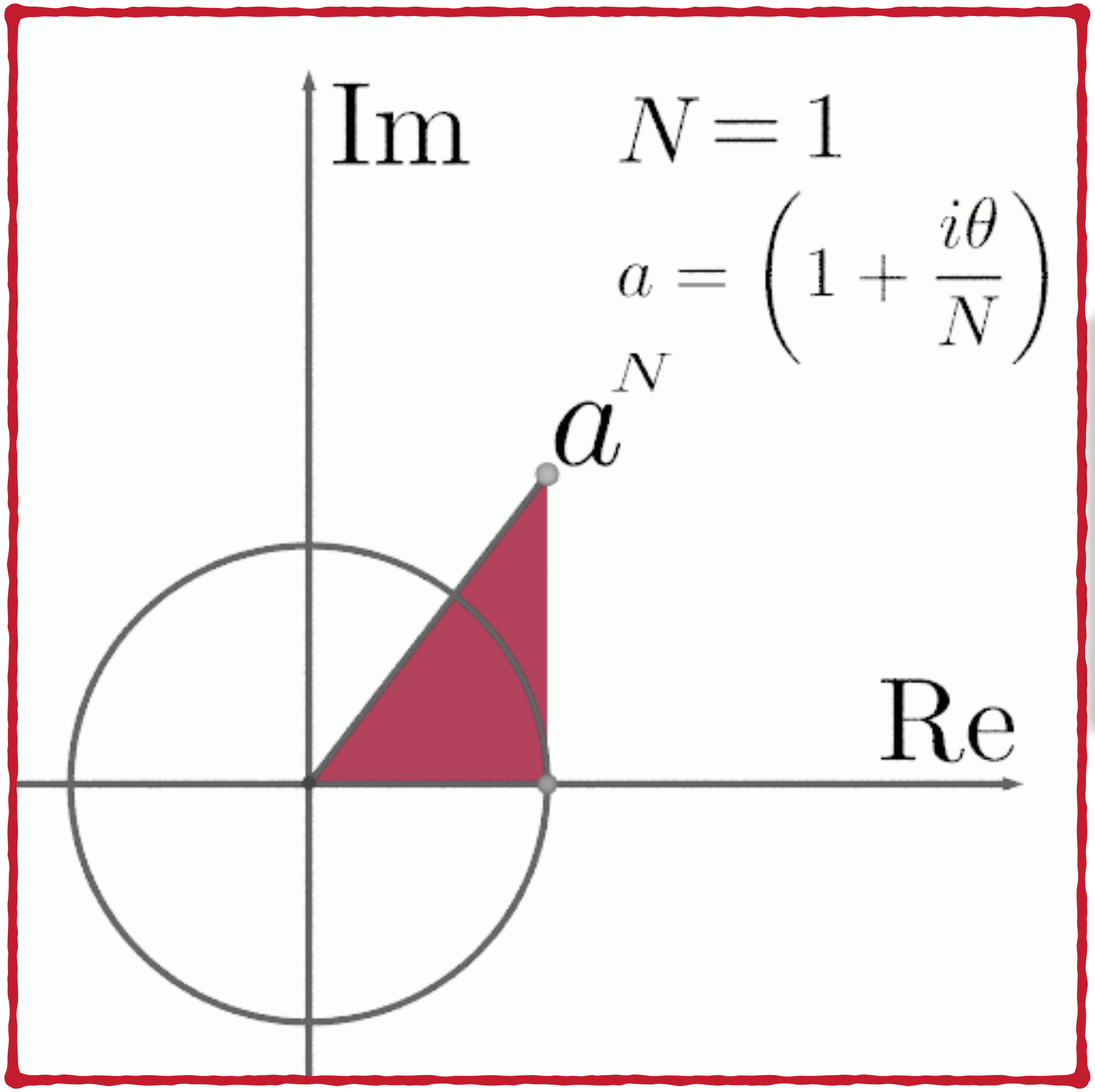
$$e^z = \lim_{N\to\infty} \left(1 + \frac{z}{N}\right)^N$$

$$e^{i\theta} = \lim_{N\to\infty} \left(1 + \frac{i\theta}{N}\right)^N = \lim_{N\to\infty} a^N = \cos\theta + i\sin\theta$$

$$e^{i\tau} = 1$$

http://hyrodium.tumblr.com/post/106436002954/euler2

ANOTHER LIMIT EXPLANATION GRAPHICALLY



$$e^{z} = \lim_{N\to\infty}\left(1+\frac{z}{N}\right)^{N}$$

$$e^{i\theta} = \lim_{N\to\infty}\left(1+\frac{i\theta}{N}\right)^{N} = \lim_{N\to\infty} a^{N} = \cos\theta + i\sin\theta$$

$$e^{i\tau} = 1$$

http://hyrodium.tumblr.com/post/106436002954/euler2

COMPLEX NUMBER REPRESENTATIONS OF SINUSOIDS

Euler's formula: $e^{ix} = cos(x) + isin(x)$

In-class Exercise:

Using Euler's formula, can you re-express cos(x) and sin(x) in terms of e, i, and x?

COMPLEX REP. OF FOURIER SERIES

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi i n t}$$

$$C_n = \frac{1}{T} \int_0^T e^{-2\pi i n t} f(t)\,dt$$

COMPLEX REP. OF FOURIER SERIES

$$f(t) = \sum_{n=-N}^{N} C_n e^{2\pi int}$$

$$C_n = \frac{1}{T}\int_0^T e^{-2\pi int} f(t)\,dt$$

See Who is Fourier Chs. 10,11
Osgood notes Sec 1.6.1